Factorization Algebras in Representation theory

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To all the kind people in my life:
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Contents

Introduction

Factorization algebras are algebraic structures designed to model observables in classical and quantum field theories. They were initially introduced in an algebrogeometric setting by Beilinson and Drinfeld on their work on conformal field theory under the name of chiral factorization algebras [BDD04]. Later using the language of ∞-categories a topological version of factorization algebras [Lur09a, AF15, GTZ14], which is the one we will consider in this work. Between the many models for observables in quantum field theories, factorization algebras have proven to be particularly useful, as can be seen from the seminal work by Costello and Gwilliams, where they are a central object for a rigorous approach to perturbative renormalization of quantum field theories [Cos11, AF15].

Factorization algebras describe quantum field theories in similar way to (co)sheaves describe manifolds or schemes, namely they describe local assignments of the theory and allow to "glue" these assignments via a local-to-global property. Motivated by the BV-BRST formalism [BV84, ASZK97], it is most convenient to consider *derived* observables in a quantum field theory. Thus it is standard to consider factorization algebras taking values in an ∞-category, in the physics literature the ∞-category theory in consideration is usually the one arising from $\mathrm{Ch}(\mathrm{Vec}_{\mathbb{C}})$ and quasisomorphisms. In general, the structure of a factorization algebra on an arbitrary ∞-category is intractable. However the subclass of locally constant factorization algebras are accessible objects that can be studied as algebras over an ∞-operad. To be precise, as algebras over the ∞-operad of little n-disks, known as $\mathbb{E}_n$-algebras. An $\mathbb{E}_1$-algebra is the same as an $\mathbb{A}_\infty$-algebra, that is, a homotopy associative algebra. On the other extreme $\mathbb{E}_\infty$-algebras are homotopy commutative algebras. Meanwhile $\mathbb{E}_n$-algebras are intermediate structures describing higher levels of commutativity between $\mathbb{E}_1$-algebras and $\mathbb{E}_\infty$-algebras.

$\mathbb{E}_n$-algebras	$n = 1$	$1 < n < \infty$	$n = \infty$
	Homotopy Associative	**Intermediate commutativity**	**Homotopy commutative**

The structure theory of $\mathbb{E}_n$-algebras has been well studied in the case of $\mathrm{Ch}(\mathrm{Vec}_{\mathbb{C}})$ and Top, for example, May's recognition theorem characterizes (grouplike) $\mathbb{E}_n$-algebras on Top as n-fold iterated loop spaces [May]. Meanwhile in the case of $\mathrm{Ch}(\mathrm{Vec}_{\mathbb{C}})$ there is extensive literature studying the relationship between $\mathbb{E}_n$-algebras and higher Poisson structures (see for example [Kel06] and references therein). However, beyond the cases of $\mathrm{Ch}(\mathrm{Vec}_{\mathbb{C}})$ and Top the structure of $\mathbb{E}_n$-algebras on other ∞-categories has been hardly studied.

One first goal of the present thesis is to give a brief introduction to the theory of $(\infty, 1)$-categories and ∞-operads and use this framework to study the structure of $\mathbb{E}_n$-algebras on weak m-categories for $m \leq 2$. To be precise we study $\mathbb{E}_n$-algebras in the bicategory of categories, and the tricategory of bicategories. The result and conjectures in this thesis can be summarized in the Table 1.

	$\mathbb{E}_1$-algebras	$\mathbb{E}_2$-algebras	$\mathbb{E}_3$-algebras	$\mathbb{E}_4$-algebras
Sets	Monads	Commutative monads	Commutative monads	Commutative monads
$\mathrm{Cat}_{(2,1)}$	Monoidal categories	Braided monoidal categories	Symmetric monoidal categories	Symmetric monoidal categories
$\mathrm{Gray}_{(3,1)}$	Monoidal 2-categories	Braided monoidal 2-categories	Sylleptic monoidal 2-categories	Symmetric monoidal 2-categories

Table 1: Summary of $\mathbb{E}_n$-algebras for weak m-categories for $m \leq 2$.

We remark that the fist two rows were already described in the literature, for example [Lur09a] and [SJ93]. Meanwhile, the last row was known as a folk result without a rigorous proof in the setting of $(\infty, 1)$-categories.

Monoidal and braided monoidal categories are important objects in different areas of mathematics. For instance in representation theory, many categories of representations enjoy these additional structures, for example $\mathrm{Rep}_{\mathrm{fd}}(A)$ the category of finite dimensional representation of an Hopf algebra is a monoidal category with the tensor product structure. Meanwhile interesting non trivial examples of braided monoidal categories arise as the representation category of quasitriangular Hopf algebras, in particular, quantum groups [Kas12]. Braided monoidal categories are also important in topology since modular tensor categories, i.e. braided monoidal categories with additional data and dualizability constraints, are the building blocks for many 3d TQFTs like the Turaev-Viro TQFT (using spherical tensor categories) and the Reshetikhin-Turaev (using modular tensor categories) [BDSPV15, KJB10, Tur92]. The role of braided monoidal bicategories in topology and representation theory is expected to be analogous to the roles of its lower dimensional analogues. From the representation theory side, many interesting braided monoidal categories should arise either as 2-representations of Hopf 2-algebras [SD97], or as 2-categorifications of the representations category of quasi-triangular Hopf algebras. From the topology side, the rough expectation is that braided monoidal categories, together with additional structure and dualizability constraints, can be used to construct extended 4d TQFTs [BN96]. However, despite the mentioned motivation and expected roles of braided monoidal bicategories in mathematics, explicit examples of them are still scarce in the literature.

The second goal of the present work is to lay the higher categorical foundations for the development of explicit examples of $\mathbb{E}_n$-algebras on bicategories. We expect that the 2-categories arising from categorification in representation theory may be endowed with interesting $\mathbb{E}_n$-algebra structures. Let us take a moment to explain the motivation: In [BFK00] the tensor product of the fundamental representations of $U(\mathfrak{sl}_2)$ was categorified via (derived) projective and Zuckerman functors on the category of maximal parabolics of $\mathcal{O}(\mathfrak{sl}_n)$. Later the introduction of the graded version of maximal parabolic category $\mathcal{O}$ in [FKS07] generalized the construction to the fundamental representations of $U_q(\mathfrak{sl}_2)$ (for q generic). Since the category generated by tensor products of the natural representation of the quantum group $U_q(\mathfrak{sl}_2)$ is braided monoidal, in the light of the previous discussion, it could be hoped that the 2-category of graded maximal parabolic categories, projective functors and natural transformations could be endowed with the structure of a $\mathbb{E}_2$-algebra. We will consider a simplification of this category in the following way: Since a parabolic category is a finite abelian category, it can be realized as modules over an associative algebra. Moreover, since projective functors are exact and preserve coproducts, then Eilenberg-Watts theorem implies that projective functors and isomorphism between them can be presented by projective bimodules and bimodule isomorphisms. The work in [BS08, BS10, BS11] shows that the algebras and bimodules in consideration are generalizations of Khovanov arc algebras and geometric bimodules. Proceeding in an *ad-hoc* way we hope that both, the bicategory of maximal parabolics and projective functors and the bicategory of geometric bimodules, are possible examples of interesting $\mathbb{E}_n$-algebras on Bicat.

Structure of the Thesis

In the first chapter we motivate $(\infty, 1)$-categories and introduce the main framework of quasicategories, which is the model of $(\infty, 1)$-categories that we will use as the backbone for the latter chapters. Quasicategories are a model for $(\infty, 1)$-categories based on simplicial sets, they are precisely simplicial sets satisfying an inner horn lifting condition, and they naturally generalize both categories and topological spaces. We review the main properties of quasicategories and generalizations of the classical definition in category theory to the framework on quasicategories. This chapter does not contain anything new and most of its content may be found in the first chapters of [Lur09b] or [Rez21]. Nevertheless, we opted to add it to this work in order to make it as self-contained as possible. The reader comfortable with quasicategories may skip this chapter.

In the second chapter we describe the main examples of quasicategories for our work. We review the definition of topological categories, bicategories and tricategories. In the case of bicategories we describe the higher analogues of functors and natural transformations, namely: pseudofunctors, pseudonatural transformations and modifications. Moreover, we present our main example of a tricategory, the tricategory of all bicategories, pseudofunctors, pseudonatural transformations and modifications. For each of the higher categories described in this chapter we present an associated nerve construction which, under appropriate conditions, will give an example of a quasicategory. Using the results of [Dus02] and [Car15] we characterize precisely the conditions on bicategories and tricategories required, in order that its nerves are quasicategories. The important results of this section can be summarized in the following theorems:

Theorem 0.0.1. Let $\mathrm{Cat}_{(2,1)}$ be the 2-category of categories, functors and invertible natural transformations. Then the bicategory nerve $\mathbb{C}\mathrm{at}^{\times}_{(2,1)} := N_2(\mathrm{Cat}_{(2,1)})$ of $\mathrm{Cat}_{(2,1)}$ is a quasicategory.

Theorem 0.0.2. Let $\mathrm{Gray}_{(3,1)}$ be the 2-category of 2-categories, 2-functors, adjoint equivalences, and invertible modifications. Then the tricategory nerve $\mathbb{G}\mathrm{ray}_{(3,1)} := N_3(\mathrm{Gray}_{(3,1)})$ of $\mathrm{Gray}_{(3,1)}$ is a quasicategory.

In the third chapter we will introduce the theory of ∞-operads as developed by Lurie, which we call quasioperads [Lur12, Lur09a]. In this chapter we take the opportunity to motivate, from a physics perspective, the definition of factorization algebras and their natural description as algebras over an operad. We introduce the basic theory of (strict) operads and algebras over them. We present various examples of operads and their algebras. Based on the intuition of strict operads, we motivate the definition of a quasioperad, symmetric monoidal quasicategories and algebras over quasioperads. Moreover we introduce the Cartesian symmetric monoidal quasicategory, which will allow to consider explicit examples of algebras over quasioperads. Most of this chapter can be found in classical texts on operads [LV12, MSS02] and on Lurie's treatment on ∞-operads [Lur09b]. It serves mainly for motivation and to establish the framework for chapter 4.

In the fourth chapter we review the main properties of $\mathbb{E}_n$-algebras and give the main results of this thesis. We begin by briefly sketching the equivalence between locally constant factorization algebras and $\mathbb{E}_n$-algebras. Then we explain precisely how $\mathbb{E}_n$-algebras model intermediate commutativity between homotopy associative algebras and homotopy commutative algebras. Additionally we discuss the stabilization hypothesis, which states that an $\mathbb{E}_n$-algebra on a weak $(m,1)$-category is homotopy commutative for $n > m$. Later, building on the results of the previous chapters, we study the structure of $\mathbb{E}_n$-algebras on the symmetric monoidal quasicategory of $\mathrm{Cat}_{(2,1)}$ and give a detailed proofs for the following theorems:

Theorem 0.0.3. An $\mathbb{E}_1$-algebra in the symmetric monoidal quasicategory $\mathbb{C}\mathrm{at}^{\times}_{(2,1)}$ determines and is determined by a monoidal category.

Theorem 0.0.4. An $\mathbb{E}_2$-algebra in the symmetric monoidal quasicategory $\mathbb{C}\mathrm{at}^{\times}_{(2,1)}$ determines and is determined by a braided monoidal category.

Furthermore, using the same techniques we extend the result to characterize $\mathbb{E}_1$-algebras and $\mathbb{E}_2$-algebras in the symmetric monoidal quasicategory of $\mathrm{Gray}_{(3,1)}$. We also give some ideas towards a characterization of $\mathbb{E}_3$-algebras. Explicitly, we give proofs for the following theorems and work towards the subsequent conjecture:

Theorem 0.0.5. An $\mathbb{E}_1$-algebra in the symmetric monoidal quasicategory $\mathbb{G}\mathrm{ray}^{\times}_{(3,1)}$ determines and is determined by a monoidal 2-category $\mathcal{B}$.

Theorem 0.0.6. An $\mathbb{E}_2$-algebra in the symmetric monoidal quasicategory $\mathbb{G}\mathrm{ray}^{\times}_{(3,1)}$ determines and is determined by a braided monoidal 2-category $\mathcal{B}$.

Conjecture 0.0.7. An $\mathbb{E}_3$-algebra in the symmetric monoidal quasicategory $\mathbb{G}\mathrm{ray}^{\times}_{(3,1)}$ determines and is determined by a sylleptic monoidal 2-category $\mathcal{B}$.

The fifth chapter is an outlook on examples of $\mathbb{E}_n$-algebras that appear in representation theory. First, we consider the full subcategory $\mathrm{Fund}(U_q(\mathfrak{sl}_2))$ of $\mathrm{Rep}_{\mathrm{fd}}(U_q(\mathfrak{sl}_2))$ generated by tensor products of the fundamental representations of $U_q(\mathfrak{sl}_2)$ and describe its structure as a braided monoidal category. This is a particular case of a general fact: the representation category of a quasitriangular Hopf algebra is a braided monoidal category (see [DM82]). Then we will present two 2-categories associated to the categorifications of tensor products of the fundamental representation of $U_q(\mathfrak{sl}_2)$: first, a 2-category build from singular blocks of $\mathcal{O}(\mathfrak{sl}_n)$ and projective functors (see [BFK00]); second the 2-category of graded Geometric bimodules over Khovanov arc algebras (see [BS11]). From the categorification intuition and the fact $\mathrm{Fund}(U_q(\mathfrak{sl}_2))$ is a $\mathbb{E}_2$-algebra in $\mathbb{C}\mathrm{at}$, we hope that these two 2-categories can be endowed with the structure of $\mathbb{E}_n$-algebras on $\mathbb{G}\mathrm{ray}$. The construction of $\mathbb{E}_n$-algebras structures on the previous 2-categories will be postponed to a subsequent work.

Chapter 1

The Idea behind $(\infty, 1)$-Categories and Quasicategories

Since its introduction in the 1940s, category theory has proven to be an essential unifying framework in mathematics allowing to identify the essential structures in mathematics. However, in the presence of homotopy or weak equivalences, the traditional concepts of category theory sometimes fail to describe natural constructions from these contexts. Higher category theory is an alternative to describe these constructions by considering *higher* morphisms. Intuitively an n-category is like a category, but instead of having hom-sets between objects, it has homs between any two objects given by $(n-1)$-categories, together with compositions, given by $(n-1)$-category maps, and distinguished identity objects. However, in practice, for example when considering bicategories, one may encounter that the mentioned composition and identities do not need to be associative, or satisfy the unit identities, on the nose. Instead they satisfy such features up to a $(n+1)$-morphism, which also satisfy some coherent conditions, again up to higher morphisms. Due to the increasing complexity of the coherence conditions, it is unfeasible to give a explicit description of all the higher coherence conditions. For example, the coherence conditions of a 3-category or a 4-category are already quite complicated, see for example [Tri06].

Although n-categories in general might be unwieldy it turns out that by restricting to n-morphisms that are invertible, up to higher morphisms, the situation becomes more tractable. An n-category in which all j-morphisms for $j \geq k+1$ are invertible is called a (n, k)-category. Here we use the adjective invertible in the *weak* sense: a j-morphism $f : x \to y$ is invertible if it has an inverse $g : y \to x$ for which

$$f \circ g \cong_j id_x, \quad \text{and} \quad g \circ f \cong_j id_y,$$

where $h_1 \cong_j h_2$ means h_1, h_2 are isomorphic via an invertible $(j+1)$-morphism. Notice that for $j = n$, due to the lack of higher morphisms, one does require the equalities

$$f \circ g = id_x, \quad \text{and} \quad g \circ f = id_y.$$

The upshot of considering (n, k)-categories is that there is a convenient model for the case of $(\infty, 0)$-categories. Intuitively, a topological space X models a $(\infty, 0)$-category in the following way: one may consider the points in X as objects, then morphisms between objects are given by paths in X, while higher morphisms are given by homotopies of paths, homotopies between homotopies, and so on. Notice that necessarily all the morphisms are invertible since reparametrising one may obtain the inverse path and the inverse higher homotopies. Actually, the homotopy hypothesis claims that topological spaces should model $(\infty, 0)$-categories.

Thus the start point for a model for $(\infty, 1)$-categories is the existence of a $(\infty, 0)$-category of morphisms between any two objects, i.e., there is a topological space of morphisms between any two objects together with identity morphisms and composition laws, which are required to be associative and unital up to homotopy. Moreover any model for a $(\infty, 1)$-category should allow for the basic constructions of category theory such as functors, limits and adjoints, within the context of weak equivalence/homotopy. Therefore from the previous discussion we can expect the following properties as features of any model for $(\infty, 1)$-categories:

- a set or collection of objects in the $(\infty, 1)$-category $\mathcal{C}$,
- for any two objects $x, y \in \mathcal{C}$ a $(\infty, 0)$-category of morphisms, or *morphism space* $\mathcal{C}(x, y)$,
- a homotopy category $h\mathcal{C}$, which is an ordinary category with the same object as $\mathcal{C}$ and whose morphisms correspond to equivalences of 1-morphisms modulo higher morphisms,
- constructions from usual category theory such as functors, (co)limits, adjuntions and monoidal structures.

In this chapter, following [Lur09a, Rez21] we will describe a model for $(\infty, 1)$-categories known in the literature as quasicategories. Quasicategories provide a combinatorial model for $(\infty, 1)$-categories based on the theory of simplicial sets. The idea to use simplicial sets as the framework to develop a model of $(\infty, 1)$-categories can be motivated by the fact that both topological spaces and ordinary categories are embedded as full subcategories of the category of simplicial sets. Quasicategories have been probably the most developed model for $(\infty, 1)$-categories with important contributors such as Joyal [Joy02], Lurie [Lur09b], etc. and one of its big advantages is that most constructions can be realized as particular instances of the well developed theory of simplicial sets. We will mostly follow the notation and discussion about simplicial sets and quasicategories as found in [Rez21].

1.1 Basics of Simplicial Sets

To define quasi-categories, we first introduce some basic notions and notation of the theory of simplicial sets.

Definition 1.1.1. The **simplicial indexing category** Δ is the category with objects linearly ordered sets $[n] = \{0 \leq \cdots \leq n\}$ and morphism (non-strict) increasing maps. Morphisms $\alpha : [n] \to [m]$ in Δ are also called **simplicial operators**.

A simplicial operator $\alpha : [n] \to [m]$ is characterized by the tuple of its image and we will represent them by

$$\alpha = \langle \alpha_1, \cdots, \alpha_n \rangle : [n] \to [m], \quad \text{where } \alpha_i := \alpha(i) \text{ for } i \in [n].$$

For example $\alpha = \langle 001 \rangle : [2] \to [1]$ is the map

$$0 \mapsto 0, \quad 1 \mapsto 0, \quad 2 \mapsto 1.$$

It is a fact (see [Fri12, Definition 3.2]) that any simplicial operator can be written as a composition of distinguished simplicial operators known as **face** and **degeneracy operators**:

$$d_i := \langle 0, \ldots, \hat{i}, \cdots, n\rangle : [n-1] \to [n]$$
$$s_i := \langle 0, \ldots, i, i, \cdots, n\rangle : [n+1] \to [n], \quad 0$$

satisfying the simplicial identities $s_i \circ s_j = s_{j-1} \circ s_i$ if $i < j$, $d_i \circ d_j = d_j \circ d_{i-1}$ if $i > j$ and

$$s_j \circ d_i = \begin{cases} d_i \circ s_{j-1} & \text{if } i < j, \\ \text{id} & \text{if } i \in \{j, j+1\}, \\ d_{i-1} \circ s_j & \text{if } i > j+1. \end{cases}$$

Definition 1.1.2. A **simplicial set** is a functor $\mathcal{C} : \Delta^{op} \to \text{Set}$. Explicitly this is given by the image of the objects $C_n := \mathcal{C}[n]$, which we call **n-simplices**, and the image of the simplicial operators $\mathcal{C}\alpha : C_n \to C_m$ for $\alpha : [m] \to [n]$ which, abusing notation, we also call **simplicial operators**.

Since morphism in Δ generated by faces and degeneracies, then one may define a simplicial set by the assignments C_n and the image of the faces and degeneracies $\mathcal{C}d_i$ and $\mathcal{C}\partial_i$. It is typical to consider the simplicial operators as *acting* on the right on the sets C_n (since a simplicial set is a contravariant functor) and write

$$c\alpha := \mathcal{C}\alpha(c) \quad \text{for } \alpha : [n] \to [m] \text{ and } c \in C_m,$$

with this notation we can summarize the condition of $\mathcal{C}$ being a functor as $(c\alpha)\beta = c(\alpha\beta)$ and $c\,(\text{id}) = c$. Moreover adding to the introduced notation of simplicial operators we introduce the shorthand

$$c_{\alpha_1, \cdots, \alpha_n} := c\langle \alpha_1, \cdots \alpha_n\rangle.$$

For example, for $c \in C_0$ we denote $c_{00} \in C_1$ the image of c under the simplicial operator $\langle 00\rangle$.

Definition 1.1.3. A **homorphism or map** between simplicial sets X, Y is a natural transformation, i.e. an assignment $\tau_n : X_n \to Y_n$ for every n compatible with the simplicial operators on X and Y. The set of homorphisms between two simplicial sets is denoted $\text{Hom}_{\text{sSet}}(X, Y)$. Simplicial sets together with their homorphisms form a category denoted sSets.

Example 1.1.4. The **standard n-simplex** Δ^n is the simplicial set represented by the object $[n]$, i.e.

$$\Delta^n := \text{Hom}_\Delta(-, [n]).$$

By the Yoneda lemma it holds

$$\text{Hom}_{sSet}(\Delta^n, X) \cong X_n; \quad f \mapsto f(\text{id}_{\Delta^n}).$$

By the virtue of their (co)representability we can give some pictures of the standard simplices by a picture of the category $[n]$. For example for low n these look like:

Δ^0 :

$$\langle 0\rangle$$

Δ^1 :

$$\langle 0\rangle \xrightarrow{\langle 01\rangle} \langle 1\rangle$$

Δ^2 :

$$\begin{tikzcd}
 & \langle 1\rangle \arrow[dd, "\langle 12\rangle"] \\
\langle 0\rangle \arrow[ur, "\langle 01\rangle"] \arrow[dr, "\langle 02\rangle"'] & \\
 & \langle 2\rangle
\end{tikzcd}$$

Δ^3 :

$$\begin{tikzcd}
 & \langle 1\rangle \arrow[dd] \arrow[dr, "\langle 13\rangle"] & \\
\langle 0\rangle \arrow[ur, "\langle 01\rangle"] \arrow[dr, "\langle 02\rangle"'] \arrow[rr, "\langle 03\rangle" description] & & \langle 3\rangle \\
 & \langle 2\rangle \arrow[ur, "\langle 23\rangle"'] &
\end{tikzcd}$$

By the Yoneda embedding the $\langle n\rangle$ represent the 0-simplices of Δ^n and the arrows $\langle ij\rangle$ represent 1-simplices. For aesthetic reasons we do not explicitly draw the higher simplices, but these should be considered as fillings of the above images in the right dimensions. These pictures become more relevant if one considers their geometric counterpart. For each of the pictures above one can consider the **topological standard simplices**

$$|\Delta|^n = \{(x_0,\dots,x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1, \quad 0 \le x_i \le 1\},$$

which actually looks like "fillings" of the above pictures. Thus $|\Delta^0|$ is a point, $|\Delta^1|$ is a line, $|\Delta^2|$ is a filled triangle, $|\Delta^3|$ a filled tetrahedron, etc.

Example 1.1.5. The **n-boundary of** Δ^n is the subsimplicial set (i.e. the subfunctor) $\partial\Delta^n \subset \Delta^n$ defined by

$$\partial\Delta^n = \bigcup_{i=1,\dots,n} \Delta^{[n]\setminus i} \subset \Delta^n,$$

where $\Delta^{[n]\setminus i}$ denotes the simplicial set corepresented by the ordered set $[n]\setminus i$.

The **i-horn of** Δ^n is the subsimplicial set defined by $\partial\Delta^n$ minus the i-th face, that is

$$\Lambda^n_i = \bigcup_{j\neq i} \Delta^{[n]\setminus j} \subset \Delta^n.$$

If $0 < i < n$ then we call Λ^n_i an **inner horn**. Just as in the case of simplices one can visualize horns. For instance, in the case $n = 2$, the horns look like

Λ^2_0 :

$$\begin{tikzcd}
 & \langle 1\rangle & \\
\langle 0\rangle \arrow[ur, "\langle 01\rangle"] \arrow[rr, "\langle 02\rangle"'] & & \langle 2\rangle
\end{tikzcd}$$

Λ^2_1 :

$$\begin{tikzcd}
 & \langle 1\rangle \arrow[dr, "\langle 12\rangle"] & \\
\langle 0\rangle \arrow[ur, "\langle 01\rangle"] & & \langle 2\rangle
\end{tikzcd}$$

Λ^2_2 :

$$\begin{tikzcd}
 & \langle 1\rangle \arrow[dr, "\langle 12\rangle"] & \\
\langle 0\rangle \arrow[rr, "\langle 02\rangle"'] & & \langle 2\rangle
\end{tikzcd}$$

Again here the topological picture is useful, for example, in the case $n = 3$ the horn Λ^3_i actually looks like a *real life* horn, a hollow tetrahedron without one face.

Being a category of functors, limits and colimits in sSets are computed pointwise. Thus for instance the **Cartesian product** of simplicial sets X and Y is the simplicial set $X \times Y$ with

$$(X \times Y)_n := X_n \times Y_n, \quad \text{and} \quad (x, y)\alpha \times \beta = (x\alpha, y\beta).$$

We will use the same notation used as in Sets to refer to (co)limits such as products, coproducts, pullbacks, pushouts, etc. Simplicial sets can be further endowed with a closed category structure, i.e. there exist an enrichment over itself satisfying appropriate adjuntion properties. To be precise given simplicial sets X and Y, we construct a **hom-simplicial set** which on objects is given by

$$\underline{\text{sSets}}(X, Y)_n := \text{sSets}(\Delta^n \times X, Y),$$

and the simplicial operators are induced from simplicial operators in Δ. Note that the 0-simplicies in the enriched hom-simplicial set correspond to the morphism sets in the original category.

Lemma 1.1.6. The enrichment satisfies the adjunction properties

$$\text{sSets}(X \times Y, Z) \xrightarrow{\sim} \text{sSets}(X, \underline{\text{sSets}}(Y, Z)),$$

and

$$\underline{\text{sSets}}(X \times Y, Z) \cong \underline{\text{sSets}}(X, \underline{\text{sSets}}(Y, Z)),$$

where this last one is understood as an isomorphism of simplicial sets

Proof. See [Rez21, Propositio 13.4]. □

1.2 Quasicategories: Definition and First Properties

Definition 1.2.1. A **quasicategory** is a simplicial set $\mathcal{C}$ which satisfies the inner horn extension property, i.e. for each $0 < i < n$ and every map of simplicial sets $f : \Lambda^n_i \to \mathcal{C}$

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

there exist a morphism a dotted arrow making the diagram commutative.

Before discussing the properties of quasicategories we present one of the motivating examples for the theory: The quasicategory coming from a category.

Definition 1.2.2. Let C be a small category. The nerve of a category $N(C)$ is the simplicial set given by the set of n-composable morphisms. That is, if we denote by C_0, C_1 the objects and morphisms in C, respectively, and $s(f), t(f)$ the source and target of a morphism f then

$$\begin{aligned} (N\mathcal{C})_n &:= \left\{ (g_1, \cdots, g_n) \in (C_1)^{\times n} | \, t(g_{i-1}) = s(g_i)) \right\} \\ &= C_1 \times_{C_0} C_1 \times_{C_0} \cdots \times_{C_0} C_1 \quad (n \text{ times}), \end{aligned}$$

where the pullbacks above are considered over the morphism s, t. The simplicial operators are generated by the face and degeneracy maps

$$s_i : (N\mathcal{C})_n \to (N\mathcal{C})_{n+1}, \quad (g_1, \ldots, g_n)s_i = (g_1, \ldots, g_{i-1}, \mathrm{id}, g_i, \ldots, g_n),$$

$$d_i : (N\mathcal{C})_n \to (N\mathcal{C})_{n-1}, \quad \begin{cases} (g_1, \ldots, g_n)d_1 = (g_2, \ldots, g_n), \\ (g_1, \ldots, g_n)d_i = (g_1, \ldots, g_i g_{i+1}, g_n) \quad for\ 1 < i < n, \\ (g_1, \ldots, g_n)d_n = (g_1, \ldots, g_{n-1}). \end{cases}$$

Example 1.2.3. We will give an example to enlighten the meaning of the inner horn lifting condition. Consider the category Q with 4 objects $\{1, 2, 3, 4\}$ and one morphism $i \to j$ when $i \leq j$

$$\bullet_0 \to \bullet_1 \to \bullet_2 \to \bullet_3.$$

Let first consider the lifting condition for inner 2-horns. A general map of simplicials sets $f : \Lambda^2_1 \to N(Q)$ looks like

$$\begin{tikzcd} & \langle 1 \rangle \arrow[dr, "\langle 12 \rangle"] & & & & \bullet_j \arrow[dr] & \\ \langle 0 \rangle \arrow[ur, "\langle 01 \rangle"] & & \langle 2 \rangle & \mapsto & \bullet_i \arrow[ur] & & \bullet_k \end{tikzcd},$$

where $i \leq j \leq k$. A lifting of the inner horn is given by the composable sequence $\bullet_i \to \bullet_j \to \bullet_k$. Notice that lifting to the horn and the restricting to the the edge $\langle 0 \rangle \to \langle 2 \rangle$ determines the composition $\bullet_i \to \bullet_k$ in Q. Thus we may regard the lifting a inner 2-horn as specifying the composition in our category Q.

Lets now showcase the necessity to restrict ourselves to inner horns. Consider the image the map $P_0 : \Lambda^2_0 \to N(Q)$ given by

$$\begin{tikzcd} & \langle 1 \rangle & & & & \bullet_1 & \\ \langle 0 \rangle \arrow[ur, "\langle 01 \rangle"] \arrow[rr, "\langle 02 \rangle"'] & & \langle 2 \rangle & \mapsto & \bullet_0 \arrow[ur] \arrow[rr, "id"'] & & \bullet_0 \end{tikzcd}$$

A lifting of P_0 to a map $P : \Delta^2 \to N(Q)$ would imply the existence of a map $\bullet_1 \to \bullet_0$, which does not exist. As we will see, the lifting property for inner horns will be related to the existence of inverses *up to homotopy*.

At last, we will consider the maps the lifting condition of 3-horns. Consider a map of simplicial sets $f_i : \Lambda^3_i \to N(Q)$, the image of such a map is characterized by the images of the faces $\Delta^{[3]\setminus\{i\}}$ opposite to the vertex $\langle i \rangle$ (Figure 1.1). Note that the images of the faces $\Delta^{[3]\setminus 0}$ and $\Delta^{[3]\setminus 3}$ are the only collection of composable paths that do not skip a vertex. Thus for an inner horn (which contains both $\Delta^{[3]\setminus 0}$ and $\Delta^{[3]\setminus 3}$ as faces) any path that skips a vertex is the composition of adjacent paths (paths of lenght 1). To define a lifting out of an inner horn $\Lambda^{3\setminus[i]}$ to Δ^3 is the same as giving a collection of 3 composable paths extending the image of the inner horn.

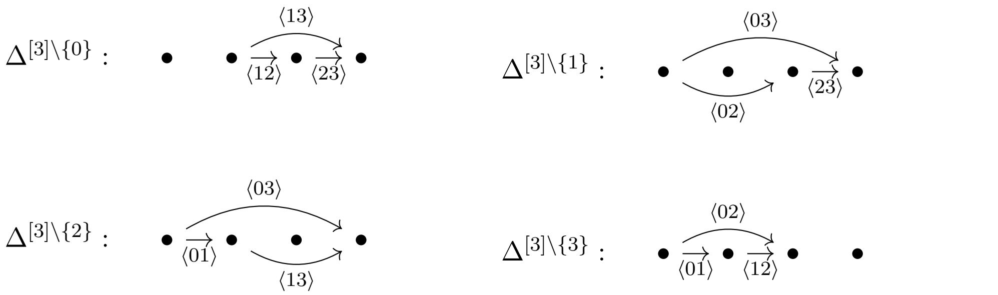


Figure 1.1: Images of the faces $\Delta^{[3]\setminus i}$ on the category Q. A Horn Λ^n_i is the union of 3 of these diagrams (omitting the face $\Delta^{[3]\setminus\{i\}}$).

In the case of an inner horn, since every morphism is the composition of adjacent paths, we can give a lift of the horn by the following composable sequence of paths

$$\bullet \xrightarrow{\langle 01\rangle} \bullet \xrightarrow{\langle 12\rangle} \bullet \xrightarrow{\langle 23\rangle} \bullet$$

Note that the argument does not work if we try lo lift an outer horn. For example for the outer horn Λ^3_0 there is not a witness that $\langle 12\rangle$ and $\langle 23\rangle$ are composable paths (since such a witness exist just in $\Delta^{[3]\setminus\{0\}}$ which is not a face in Λ^3_0).

We can use the intuition of the above example to prove that the nerve of a category is indeed a quasicategory.

Theorem 1.2.4. Let C be a small category. Then the nerve $N(C)$ of C is a quasicategory.

Proof. We want to see than any inner horn map $P_k : \Delta^n_k \to N(C)$ extends to a map $P : \Delta^n \to N(C)$. As a visualisation aid consider explicitly the Figure 1.1. Restricting to the each of the 1-simplices of the horn $\Delta^{i,i+1}$ we obtain paths $p_i : v_i \to v_1$ which can be organized into n composable paths

$$v_0 \xrightarrow{p_{0,1}} v_1 \xrightarrow{p_{1,2}} \cdots \xrightarrow{p_{n-2,n-1}} v_n \xrightarrow{p_{n-1,n}} v_{n+1},$$

By the Yoneda lemma this can be seen as a map $P : \Delta^n \to N(C)$, which we claim is an extension of the inner horn.

Indeed, the restriction to the $n-1$ dimensional faces $\Delta^{[n]\setminus j}$ of P, which we will denote P_j, are determined by $(n-2)$-composable maps on the $n-1$ vertices $\{v_i\}_{0\leq i\neq j\leq n}$. Any such collection of composable paths are either paths between adjacent vertices $v_i \to v_{i+1}$ or are paths skipping a vertex $v_i \to v_i + 2$. We are essentially in the same picture as in the example 1.2.2, that is, these paths are either a path of adjacent edges $v_i \xrightarrow{p_i} v_{i+1}$ or they are determined by the restriction of P_k to the 2-simplex $\Delta^{i,i+1,i+2}$. In the latter case, since the horn Λ^n_k was taken to be inner, the same argument as in example 1.2.2 implies it must be the composition of the paths $v_i \xrightarrow{p_i} v_{i+1} \xrightarrow{p_{i+1}} v_{i+2}$. We realize that in both cases the paths in question agree with the restriction of P, thus the proposed lifting indeed extends the inner horn. $\square$

Objects, Morphisms and Homotopies in Quasicategories

For a quasicategory $\mathcal{C}$ we may think of the 0-simplices as objects and 1-simplices as morphisms. To be precise for $x, y \in \mathcal{C}_0$ we will think of the set

$$\hom_{\mathcal{C}}(x,y) = \{f \in \mathcal{C}_1 \mid f_0 = x,\ f_1 = y\},$$

as the set of *morphisms* from x to y in $\mathcal{C}$. We will denote by id_x the element $x_{00} \in \hom_{\mathcal{C}}(x,x)$ and regard it as the *identity* on x. Given $f \in \hom_{\mathcal{C}}(x,y)$ and $\hom_{\mathcal{C}}(y,z)$ a **composite** of f, g is a $h \in \hom_{\mathcal{C}}(x,z)$ such that there exist a 2-simplex $a \in \mathcal{C}_2$ with

$$a_{01} = f, \quad a_{12} = g, \quad a_{02} = h, \qquad \text{pictorially} \quad \begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "a"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"] & {} & z \end{tikzcd}$$

The defining property of quasicategories assures composites always exist, since the pair (f,g) defines an inner horn which can be filled. However composites are not unique since there may exist many different horn fillings. Nevertheless, as we will see, all composites are *homotopic* in a precise sense.

Definition 1.2.5. Let $\mathcal{C}$ be a quasicategory. We say that 1-simplices $f, g \in \hom_{\mathcal{C}}(x,y)$ are **homotopic**, and denote it by $f \sim g$, if there exist a $a \in \mathcal{C}_2$ such that $a_{01} = id_x, \quad a_{02} = f, \quad a_{12} = g$ or there exist $b \in \mathcal{C}_2$ such that $b_{01} = id_x$, $b_{02} = f$, $b_{12} = g$; pictorially

$$\begin{tikzcd} x \arrow[dr, "f"] \arrow[d, "id_x"'] & \\ x \arrow[r, "g"'] \arrow[r, phantom, "a", shift left=3] & y \end{tikzcd} , \qquad \begin{tikzcd} & y \arrow[d, "id_x"] \\ x \arrow[ur, "g"] \arrow[r, "f"'] \arrow[r, phantom, "b", shift left=3] & y \end{tikzcd}$$

Furthermore, we say a 1-simplex $f : x \to y$ is an **equivalence** if there exist another 1-simplex $g : y \to x$ such that $gf \sim id_x$ and $fg \sim id_y$.

Proposition 1.2.6. Homotopy is an equivalence relation on $\hom_{\mathcal{C}}(x,y)$

Proof. Reflexiveness is exhibited by the degenerate simplex $f_{001} \in C_2$

$$\begin{tikzcd} x \arrow[dr, "f"] \arrow[d, "id_x"'] & \\ x \arrow[r, "f"'] \arrow[r, phantom, "f_{001}", shift left=3] & y \end{tikzcd}$$

On the other hand assume $f \sim g$ and $g \sim h$ are exhibited by 2-simplices $a, b \in C_2$ respectively. Consider the following diagrams

$$\begin{tikzcd} x \arrow[dr, "id_x"] \arrow[drr, bend left, "f"] \arrow[dd, bend right, "id_x"'] & & \\ {} \arrow[r, phantom, "x_{000}"] & x \arrow[r, dashed, "g"] \arrow[dl, "id_x"'] & y \\ x \arrow[urr, bend right, "h"'] & & \end{tikzcd} \qquad \begin{tikzcd} x \arrow[dr, "id_x"] \arrow[drr, bend left, "g"] \arrow[dd, bend right, "f"'] & & \\ {} \arrow[r, phantom, "a"] & x \arrow[r, dashed, "g"] \arrow[dl, "g"'] & y \\ y \arrow[urr, bend right, "id_y"'] & & \end{tikzcd}$$

which are actually inner horns in Δ^2. Therefore, by definition of a quasicategory, there exists a filling of the diagrams; transitivity and symmetry are exhibited by the opposite faces of fillings. $\square$

Definition 1.2.7. The **homotopy category** $h\mathcal{C}$ of a quasicategory is the category with objects $(h\mathcal{C})_0 := \mathcal{C}_0$ and homorphism sets given by

$$h\mathcal{C}(x, y) = \hom_{\mathcal{C}}(x, y)/\sim .$$

The composition is defined by $[f] \circ [g] = [h]$, for h a composite of (f, g), and identity morphisms $id_x \in h\mathcal{C}(x, x)$ are given by the classes of x_{00}.

We need to still prove composition is well defined, that it does not depend on the chosen composite, and that it is strictly associative and identities behave strictly as units. This is the content of the following proposition.

Proposition 1.2.8.

1. Let $f \sim f'$, $g \sim g'$ and let h be a composite of (g, f) and h' be a composite of (f', g'), then $h \sim h'$. Therefore composition in $h\mathcal{C}$ is well defined.
2. Let $f : x \to y$, then $[f] \circ [id_x] = [f] = [id_x] \circ [f]$.
3. Let $f : x \to y$, $g : y \to z$ and $h : z \to z'$ then $[(f \circ g) \circ h] = [f \circ (g \circ h)]$.

Proof. The proof is very similar to the proof of Proposition 1.2.6 and will be omitted. □

To finish this section we record some basic facts on how to construct new quasicategories from old ones.

Proposition 1.2.9.

1. Let $\mathcal{C}$ and $\mathcal{C}'$ be quasicategories, then the simplicial set $\mathcal{C} \times \mathcal{C}'$ is a quasicategory.
2. Let $\mathcal{C}$ be a quasicategory and K a simplicial set, then the simplicial $\underline{\text{sSets}}(K, \mathcal{C})$ is a quasicategory.

Proof. Property 1 is immediate from the definition. For a proof of 2 see [Rez21, Sections 19 and 20]. □

Kan complexes and Mapping Spaces

From our introductory motivation, for a given $(\infty, 1)$-category one expects that there exists an $(\infty, 0)$-category of morphisms between any given two objects, which according to the homotopy hypothesis should be modeled by a topological space. In this section we make this idea precise by introducing a the quasicategory counterpart of topological spaces, namely Kan complexes. We sketch their relationship with topological spaces, and briefly discuss why they can be considered as combinatorial models for $(\infty, 0)$-categories. Moreover for any two 0-simplices in a quasicategory we construct a Kan complex of morphisms between them.

Definition 1.2.10. A **Kan complex** is a simplicial set which has the extension property with respect to all horns Λ^n_i for $0 \le i \le n$. We denote the full subcategory of sSet of Kan complexes by Kan.

The comparison between Kan complexes and topological spaces is based on the pair of adjoint functors

$$\begin{tikzcd} \mathrm{Top} \arrow[r, shift left, "\mathrm{Sing}"] \arrow[r, phantom, "\top"] & \mathrm{sSets}, \arrow[l, shift left, "|-|"] \end{tikzcd}$$

here Sing is the simplicial chains functor and $|-|$ the geometric realization functor. Each of these categories has a notion of weak equivalence, on Top a map is a weak equivalence if it induces an isomorphism on all homotopy groups, while on sSets a morphism is a weak equivalence if the geometric realization of such morphism is a weak equivalence in the topological sense. We will denote the sets of weak equivalences in Top and sSets by $\mathcal{W}_{\mathrm{Top}}$ and $\mathcal{W}_{sSet}$, respectively. It turns out that the localizations are given by the full subcategories of CW complexes and Kan complexes in the respective cases

$$\mathrm{Top}[\mathcal{W}_{\mathrm{Top}}^{-1}] \cong \mathrm{Top}_{CW}, \qquad \mathrm{sSet}[\mathcal{W}_{\mathrm{Set}}^{-1}] \cong \mathrm{Kan}\,.$$

Moreover after localizing, the adjoint pair $|-| \dashv \mathrm{Sing}$ induces an equivalence of categories $\mathrm{Top}_{CW} \cong \mathrm{Kan}$. Thus we can conclude that, up to *weak equivalence*, topological spaces and Kan complexes model the same objects. Although giving a proof of this result is out of the aim of this work, we can nevertheless give a first proposition on this direction.

Proposition 1.2.11. Let X be a topological space and K a simplicial set. The simplicial set $\mathrm{Sing}(X)$ is a Kan complex.

Proof. By the adjunction between $|\cdot|$ and Sing we have that the following are equivalent lifting problems

$$\begin{tikzcd} \Lambda_i^n \arrow[r, "f"] \arrow[d, hook] & \mathrm{Sing}(X) \\ \Delta^n \arrow[ur, dotted, "r\circ f"'] & \end{tikzcd} \qquad \begin{tikzcd} |\Lambda_i^n|_{top} \arrow[r, "f"] \arrow[d, hook] & X \\ |\Delta^n|_{top} \arrow[ur, dotted, "r\circ f"'] & \end{tikzcd}$$

From the description of Λ_i^n as a colimit (a union) over $\Delta^{[n]\setminus\{i\}}$ the faces of Δ^n, which are isomorphic to Δ^{n-1} and the fact that geometric realization preserves colimits (its a left adjoint) we see

$$|\Lambda_i^n|_{top} = \bigcup_{j\neq i} (\Delta^n)_j = \{x \in |\Delta^n|_{top} \ \mid \ x_i = 0 \text{ for some } i \neq j\}.$$

There is continuous retraction for any horn $r : |\Delta^n|_{top} \to |\Lambda_i^n|_{top}$, thus to give a lift

$$\begin{tikzcd} |\Lambda_i^n|_{top} \arrow[r, "f"] \arrow[d, hook, shift left] & X \\ |\Delta^n|_{top} \arrow[u, bend left, "r"] \arrow[ur, dotted, "r\circ f"'] & \end{tikzcd}$$

it is enough to compose with the given retraction. $\square$

Remark 1.2.12. For the reader who is familiar with the language of model categories we can make a more precise statement (for the reader that is not familiar, but interested, in model categories we refer to [Hov07]). There are model structures on sSet (referred to as the **Kan model structure**) and Top in which the weak equivalences are described in the previous paragraph (see [Qui06]). The fibrant objects in the above models are Kan complexes and CW complexes, respectively. The pairs $|-| \dashv \mathrm{Sing}$ give a Quillen adjoint pair, thus they establish an equivalence between these model structures. The statements about localization is then a consequence of the general theory of model categories.

Back to our philosophy of $(\infty,1)$-categories one expects, by definition, that a $(\infty,1)$-category in which all 1-morphisms are invertible is a $(\infty,0)$-category. The following proposition shows exactly this in the framework of quasicategories.

Proposition 1.2.13. A simplicial set is a quasigroupoid, that is, $\mathcal{C}$ is a quasicategory such that $h\mathcal{C}$ is a groupoid, if and only if it is a Kan complex.

Proof. Assume first that $\mathcal{C}$ is a quasigroupoid and we would like to show $\mathcal{C}$ is a Kan complex. Since $\mathcal{C}$ is a quasicategory we only need to show that the horn extension property holds for the horns Λ_0^n and Λ_n^n. Let $p : \Lambda_0^n \to C$ and maps of simplicial sets

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r] \arrow[rr, bend left, "p|_{\Delta^{\{0,1\}}}"] & \Lambda_0^n \arrow[r, "p"] \arrow[d] & C \arrow[d] \\ & \Delta^n \arrow[r] \arrow[ur, dotted] & * \end{tikzcd}$$

since any morphism in C is an isomorphism, the Joyal lifting theorem ([Rez21, Theorem 32.17]) implies the horn extension property for Λ_0^n. An analogous argument shows the extension property also holds for Λ_0^n.

For the converse, note that clearly a Kan complex is a quasicategory. We will now prove $h\mathcal{C}$ is a groupoid. For this let $[f] \in h\mathcal{C}$ be represented by a $f \in C_1$, consider (not inner) horn

$$\begin{tikzcd} & y \arrow[dr, dotted, "g"] \arrow[d, phantom, "a"] & \\ x \arrow[ur, "f"] \arrow[rr, "id_x"'] & {} & z \end{tikzcd}$$

Since $\mathcal{C}$ is a Kan complex there is 2-simplex filling the horn. The homotopy class $[g]$ of opposite face to the horn g is a left inverse for $[f]$, likewise one can find a right inverse $[h]$. Since $h\mathcal{C}$ is a category, both inverses agree, and we conclude every morphism in $\mathcal{C}$ has an inverse. $\square$

Now we are in the position to introduce the mapping space between a pair of objects (0-simplices) in a quasicategory.

Definition 1.2.14. For a quasicategory $\mathcal{C}$ and objects $x, y \in \mathcal{C}_0$ the **hom-space** $\mathcal{C}(x, y)$ is the simplicial set defined by the pullback square

$$\begin{tikzcd} \mathcal{C}(x,y) \arrow[r] \arrow[d] & \underline{\mathrm{Sets}}(\Delta^1, \mathcal{C}) \arrow[d] \\ \Delta^0 \arrow[r, "(x,y)"] & \underline{\mathrm{Sets}}(\partial\Delta^1, C) \cong \mathcal{C} \times \mathcal{C}, \end{tikzcd}$$

here the right vertical morphism is induced by the inclusion $\partial\Delta^1 \hookrightarrow \Delta^1$ and the isomorphism on the bottom right corner is induced by the isomorphism $\partial\Delta^1 \cong \Delta^0 \coprod \Delta^0$. One may extend the definition to describe **n-multi hom spaces** $\mathcal{C}(x_1, \dots, x_n)$ for any set of n objects $x_1, \dots x_n \in C_0$. For this we define the quasicategory $\mathcal{C}(x_1, \dots, x_n)$ to be the pullback

$$\begin{tikzcd} \mathcal{C}(x_1, \dots, x_n) \arrow[r] \arrow[d] & \underline{\mathrm{Sets}}(\Delta^n, \mathcal{C}) \arrow[d] \\ \Delta^0 \arrow[r, "(x,y)"] & \mathcal{C}^n, \end{tikzcd}$$

The terminology of hom *space* is justified by fact that the simplicial set $\mathcal{C}(x, y)$ is a Kan complex.

Theorem 1.2.15. For any two 0-simplices in a quasicategory $\mathcal{C}$. The simplicial set $\mathcal{C}(x, y)$ is a Kan complex.

Proof. See [Rez21, Proposition 43.2]. □

The Fundamental Theorem

Having introduced mapping spaces between objects we can discuss the quasicategorical analogue of the well-known theorem in category theory: "A funtor $C \to D$ between categories is an equivalence of categories if and only if it is fully faithfull and essentially surjective". Following [Rez21] we will call this the fundamental theorem of quasicategories. For this let us first introduce the necessary definitions for quasicategories.

Definition 1.2.16. Let $\mathcal{C}$ and $\mathcal{D}$ be quasicategories, and let $F : \mathcal{C} \to \mathcal{D}$ be a map of simplicial sets.

1. F is a categorical equivalence, or just **equivalence**, if there exist another simplicial set map $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \sim id_{\mathcal{C}}$ and $F \circ G \sim id_{\mathcal{D}}$, where the homotopy is considered in the quasicategory $\underline{\text{sSets}}(F, G)$ and $\underline{\text{sSets}}(G, F)$ respectively.

2. F is **fully faithfull** if for every pair $x, y \in \mathcal{C}_0$ the map $\mathcal{C}(x, y) \to \mathcal{C}(Fx, Fy)$, induced from the universal property of the pullback, is an equivalence of Kan complexes.

3. F is essentially surjective if for every $y \in \mathcal{D}_0$ there exist an $x \in \mathcal{C}_0$ and an **equivalence** $Fx \sim y$.

Theorem 1.2.17. Let $\mathcal{C}$ and $\mathcal{D}$ be quasicategories, and let $F : \mathcal{C} \to \mathcal{D}$ be a map of simplicial sets. Then F is a categorical equivalence if and only if it is fully faithfull and essentially surjective.

Proof. See [Rez21, Section 44]. □

As expected from a hom-space we can define a composition and units on these simplicial sets. Before introducing composition, lets introduce some notation.

Definition 1.2.18.

1. The **n-spine** I^n of the n-simplex Δ^n is the subsimplicial set defined by

$$(I^n)_k := \{\ \langle a_0, \cdots, a_k \rangle \in (\Delta^n)_k \ \mid\ a_k \leq a_0 + 1\}.$$

   That is, simplex in the n-spine is a simplicial operator of the form $\langle j, j, \dots, j+1, \dots, j+1 \rangle$ where $j+1 \neq n$ (it may happen j+1 does not appear at all).

2. If $0 \leq i < j \leq n$, let $\Delta^{\{i,j\}}$ be the subsimplicial set of Δ^n isomorphic to Δ^1 with 0-simplices $\langle i \rangle$ and $\langle j \rangle$.

These subsimplicial sets can be given a picture, for example the spine in Δ^n is the graph of the longest path between $\langle 0 \rangle$ and $\langle n \rangle$, while $\Delta^{\{0,n\}}$ is the graph of the shortest path between $\langle 0 \rangle$ and $\langle n \rangle$ (see Figure 1.2 for an example in Δ^3).

I^3 :

$$\begin{tikzcd}
 & \langle 1 \rangle \arrow[dd] & \\
\langle 0 \rangle \arrow[ur] & & \langle 3 \rangle \\
 & \langle 2 \rangle \arrow[ur] &
\end{tikzcd}$$

$\Delta^{\{0,3\}}$:

$$\begin{tikzcd}
 & \langle 1 \rangle & \\
\langle 0 \rangle \arrow[rr] & & \langle 3 \rangle \\
 & \langle 2 \rangle &
\end{tikzcd}$$

Figure 1.2: A picture of I^3 and $\Delta^{\{0,3\}}$ inside Δ^3.

Let $x_1, \ldots, x_n$ be 0-simplices of $\mathcal{C}$. To define an n-composition we will consider the inclusion $I^n \subset \Delta^n$ and $\Delta^{\{0,n\}} \subset \Delta^n$ which induce maps

$$\underline{\text{sSets}}(I^n, C) \leftarrow \underline{\text{sSets}}(\Delta^n, C) \rightarrow \underline{\text{sSets}}(\Delta^{\{0,n\}}, C).$$

By taking the pullback over the point Δ^0, via the map $(x_1, \ldots, x_n) : \Delta^0 \to \mathcal{C}^n$ the above induce maps of Kan complexes

$$\mathcal{C}(x_1, x_2) \times \cdots \times \mathcal{C}(x_{n-1}, x_n) \xleftarrow{\sim} \mathcal{C}(x_1, \ldots, x_n) \rightarrow \mathcal{C}(x_1, x_n).$$

Although the first map is in the wrong direction it turns out that it is an equivalence of quasicategories, thus we may choose a weak inverse $q : \mathcal{C}(x_1, x_2) \times \cdots \times \mathcal{C}(x_{n-1}, x_n) \to \mathcal{C}(x_1, \ldots, x_n)$. Thus, up to choosing a weak inverse, there is a composition map

$$\mathcal{C}(x_1, x_2) \times \cdots \times \mathcal{C}(x_{n-1}, x_n) \rightarrow \mathcal{C}(x_1, x_n).$$

The composition is not strictly associative and depends on the weak inverse. However, any other choice of inverse induces a homotopy equivalent multiplication, and the compositions $(f \circ g) \circ h \sim f \circ (g \circ h)$ are homotopic as maps of Kan complexes. (see [Rez21, Part 7]).

Remark 1.2.19. There are many other models for hom spaces between objects, for example in [Lur09b] Lurie introduced a hom space that is strictly associative. These strictly associative hom spaces are fundamental in the comparison of quasicategories and simplicially enriched categories $\text{Cat}_{\text{sSets}}$ (see Definition 2.1.1). In brief, this comparison states that these two theories are equivalent models for $(\infty, 1)$-categories. Again, for the reader familiar with model categories we can be more precise: there are model structures in sSets and $\text{Cat}_{\text{sSets}}$, called the Joyal and Bergner model structures respectively, together with a Quillen equivalence between them. Fibrant objects in these model structures are quasicategories and Kan simplicially enriched categories, respectively. See [DS11] for a in depth discussion about mapping spaces and how they are related, and [Ber07] about the equivalence between different models for $(\infty, 1)$-categories.

1.3 Further Quasicategory Theory

In this section we introduce generalisations of usual concepts in categories theory to the framework of quasicategories, that will be used in later chapters. For example, we define subcategories, (co)limits, localization and fibrations. Each of these concepts can be easily found in the literature (for example in [Lur09b, Rez21]), nevertheless we consider stating them in this section convenient.

Subcategories of Quasicategories

In category theory one may consider the smallest subcategory having a certain set of objects, this is given by the notion of full subcategory. We want to define a similar notion for quasicategories. For this we begin by noticing that for any quasicategory $\mathcal{C}$ there is a map of simplicial sets

$$\eta : \mathcal{C} \to N(h\mathcal{C}),$$

defined on 0-simplices by the identity and on 1-simplices maps $f \to [f]$. For general n-simplices the map sends $\alpha \in \mathcal{C}_n$ to the unique sequence $(f_1, \ldots, f_n)$ of composable morphisms such that $f_i = [\alpha_{i-1,i}]$.

Definition 1.3.1. Let $\mathcal{C}$ be a quasicategory and $\mathcal{C}^0 \subset \mathcal{C}_0$ be a a subset of its objects. Let $h\mathcal{C}^0 \subset h\mathcal{C}$ be the full subcategory of $h\mathcal{C}$ spanned by the objects $\mathcal{C}^0$. Define the simplicial set $\mathcal{C}'$ as the pullback

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ N(h\mathcal{C})' \arrow[r] & N(h\mathcal{C}). \end{tikzcd}$$

Notice that by the universal property of the pullback $\mathcal{C}'$ is a quasicategory. We call $\mathcal{C}'$ the **subquasicategory** of $\mathcal{C}$ spanned by $\mathcal{C}^0$.

It is not hard to see that the simplices of the quasicategory $\mathcal{C}'$ corresponds to those simplices of $\mathcal{C}$ whose vertices lie in $\mathcal{C}^0$.

Localizations of Quasicategories

In category theory, the idea of the localization of a category C by some class W of its morphism is to consider a new category $C[W^{-1}]$ where all the morphism in S have inverses. To be precise the localization of a category C on some class W^{-1} is a map $C \to C[W^{-1}]$ with the following universal property: given another category C' and a functor $C \to D$ sending each morphism in W to an isomorphism, then there exist a unique map

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C' \\ C[W^{-1}] \arrow[ur, dashed] & \end{tikzcd}$$

Lets consider the quasicategorical analogue of such property. First, for a quasicategory $\mathcal{C}$ and simplicieal sets K and an inclusion of simplicial sets $W \subset K$, denote $\underline{sSets}^{W_{\text{iso}}}(K, \mathcal{C})$ the full quasicategory spanned by those simplicial maps $K \to \mathcal{C}$ that map every 1-simplex in K to an equivalence in $\mathcal{C}$. We state the definition of a localisation on quasicategories.

Definition 1.3.2. Let $\mathcal{C}$ be a quasicategory and $W \subset \mathcal{C}$ a subsimplicial set. A **localization of $\mathcal{C}$ with respect to W** is a pair of a quasicategory $\mathcal{C}[W^{-1}]$ and a map $l : \mathcal{C} \to \mathcal{C}[W^{-1}]$ with the following properties:

1. The map $L : \mathcal{C} \to \mathcal{C}[W^{-1}]$ sends 1-simplices in $W \subset \mathcal{C}$ to equivalences in $\mathcal{C}[W^{-1}]$.
2. For any quasicategory $\mathcal{C}'$ the map induced by precomposition with L,

$$\bar{L} : \underline{\text{sSets}}(\mathcal{C}[W^{-1}], \mathcal{C}') \to \underline{\text{sSets}}^{W_{\text{iso}}}(\mathcal{C}, \mathcal{C}'),$$

is an equivalence of categories.

We are abusing notation since $\mathcal{C}[W^{-1}]$ and L are not unique, nevertheless they are determined up to homotopy. For a proof for the existence, given by a *concrete* construction, and uniqueness up to homotopy of localisations see [Rez21, Section 53]. The notion of localization of a quasicategory will be useful when considering the equivalence between locally constant factorization algebras and $\mathbb{E}_n$-algebras.

Limits and Colimits of Quasicategories

To discuss the notion of (co)limit in quasicategories it is useful to reformulate some of the definitions used in category theory in a way that can be generalized to arbitrary simplicial sets. Let x be an object and F a functor, then a **cone** from x to F is a family of morphisms

$$\psi_X : x \to F(X),$$

that appropriately commutes with the images of morphisms under F. Cones to F form a category, and from the definition, it follows easily that the universal property of (co)limits can be rephrased as being a terminal (resp. initial) object in the category of cones. We will now present the generalization of the above picture in quasicategories.

Definition 1.3.3.

1. Let X and Y be simplicial sets. The **join** $X \star Y$ of X and Y is the simplicial set with

$$(X \star Y)_n := \coprod_{[n]=[n_1]\sqcup[n_2]} X_{n_1} \times X_{n_2},$$

where $[p] \sqcup [q]$ denotes the ordered disjoint union and $X_{-1} = \Delta^0 = Y_{-1}$. The simplicial sets $\Delta^0 \star X$ (and $X \star \Delta^0$) look like the *convex hulls* between a simplicial set and an additional vertex, and are the generalization of cones to simplicial sets.

2. Let $p : S \to X$ and $q : T \to Y$ be simplicial set maps. The **slice categories** $X_{p/}$ under p, and $X_{/q}$ over q are respectively

$$(X_{p/})_n = \mathrm{sSets}_{S/}(S \star \Delta^n, X), \quad \text{and} \quad (X_{/q})_n = \mathrm{sSets}_{T/}(\Delta^n \star T, X),$$

where $\mathrm{sSets}_{T/}$ denotes the category over T: the objects are maps of simplicial sets $T \to K$ and the morphism are $K \to K'$ making the obvious diagram commute.

We may extend the definition of the join of simplicial sets to functors

$$S \star - : \mathrm{sSets} \to \mathrm{sSets}_{S/}, \quad \text{and} \quad - \star S : \mathrm{sSets} \to \mathrm{sSets}_{T/},$$

by sending a map of simplicial sets $f : X \to Y$ to the map of simplicial sets $X \star S \to Y \star S$ determined on each n-simplex by the coproduct of the maps $id \times f_i$ for $i \leq n$. Similarly, we may extend the construction of the slices to functors

$$(p : S \to X) \mapsto X_{p/} : \mathrm{sSets}_{S/} \to \mathrm{sSets}, \quad \text{and} \quad (q : T \to X) \mapsto X_{/q} : \mathrm{sSets}_{T/} \to \mathrm{sSets},$$

by sending $f : X \to Y$ to the map of simplicial sets $X_{p/} \to Y_{p/}$ determined by postcomposition.

Proposition 1.3.4. The following are adjoint pairs.

$$
\begin{tikzcd}
\mathrm{sSets} \arrow[r, shift left, "S\star-"] \arrow[r, phantom, "\top"] & \mathrm{sSets}_{S/} \arrow[l, shift left, "\cdot_{p/}"]
\end{tikzcd}, \qquad
\begin{tikzcd}
\mathrm{sSets} \arrow[r, shift left, "-\star S"] \arrow[r, phantom, "\top"] & \mathrm{sSets}_{S/} \arrow[l, shift left, "\cdot_{/q}"]
\end{tikzcd}.
$$

Proof. See [Rez21, Section 27]. □

Definition 1.3.5. Let $\mathcal{C}$ be a quasicategory. An **initial object**, respectively a **terminal object** of $\mathcal{C}$ is an $x \in C_0$ such that every $f : \partial\Delta^n \to \mathcal{C}$ with $n \geq 1$ satisfying $f|_{\Delta^{\{0\}}} = x$, respectively $f|_{\Delta^{\{n\}}} = x$), there exists a lift $\Delta^n \to C$ making the following diagrams commute

$$
\begin{tikzcd}
\Delta^{\{0\}} \arrow[r] \arrow[rr, bend left, "x"] & \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \\
& \Delta^n \arrow[ur, dotted] &
\end{tikzcd}
\quad \text{respectively} \quad
\begin{tikzcd}
\Delta^{\{n\}} \arrow[r] \arrow[rr, bend left, "x"] & \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \\
& \Delta^n \arrow[ur, dotted] &
\end{tikzcd}
$$

Proposition 1.3.6. Let C^{init} and C^{term} be the full subquasicategories spanned by the initial, respectively terminal, objects. Then

1. The quasicategories C^{init} and C^{term} are contractible, i.e. $|C^{\text{init}}| \cong *$, and $|C^{\text{term}}| \cong *$.

2. For every $y \in \mathcal{C}_0$ the mapping space $\mathcal{C}(x, y)$, respectively $\mathcal{C}(y, x)$, is contractible.

Proof.

1. We just show it for initial objects, since the case of terminal objects is similar. The definition of initial object ensures that for any map $\partial\Delta^n \to C^{\text{init}}$ there exists a lift $\Delta^n \to C^{\text{init}}$, thus C^{init} is a Kan complex. Considering the geometric realization $|C^{\text{init}}|$, the lifting condition defining initial objects imply that $|C^{\text{init}}|$ has just trivial homotopy groups. Then, by Whitehead theorem, the space $|C^{\text{init}}|$ is contractible.

2. See [Rez21, Proposition 61.7]

□

Definition 1.3.7. Let $\mathcal{C}$ be a quasicategory, K a simplicial set and $p : K \to \mathcal{C}$ a map of simplicial sets. A p**-colimit** is an initial object of the slice category $C_{p/}$. Dually, a **p-limit** is a terminal object of $C_{/p}$. Unrolling the definition we have the following explicit descriptions:

- A p-colimit is a map $\hat{p} : K \star \Delta^0 \to \mathcal{C}$ extending p, such that, for $n \geq 1$, there exist a lift for every diagram

$$
\begin{tikzcd}
K \star \Delta^{\{0\}} \arrow[r] \arrow[rr, bend left, "\hat{p}"] & K \star \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \\
& K \star \Delta^n \arrow[ur, dotted] &
\end{tikzcd}
$$

- A p-limit is a map $\hat{p} : \Delta^0 \star K \to \mathcal{C}$ extending p, such that, for $n \geq 1$, there exist a lift for every diagram

$$\begin{tikzcd}
\Delta^{\{n\}} \star K \arrow[r] \arrow[rr, bend left, "\hat{p}"] & \partial\Delta^n \star K \arrow[r] \arrow[d] & \mathcal{C} \\
& \Delta^n \star K \arrow[ur, dotted] &
\end{tikzcd}$$

By Proposition 1.3.6 it follows that all the possible choices of limits, or colimits, for $p : K \to \mathcal{C}$ form a contractible space. Moreover, for each functor $F : \Delta^0 \star K \to \mathcal{C}$ the quasicategory $\underline{\mathrm{sSets}}(F, \hat{p})$ is a contractible Kan complex. This is the replacement of the "unique isomorphism" requirement in usual category theory in the framework of quasicategories (or $(\infty, 1)$-categories in general). We end this chapter with an example of a limit that allows to capture some of the intuition, and illustrates some consequences, of the above definition.

Example 1.3.8. 1. Let K be the simplicial set associated to the picture $\bullet \to \bullet \leftarrow \bullet$ (here the picture gives the nondegenerate simplices of K). We have $\Delta^0 \star K \cong \Delta^1 \times \Delta^1$ is the simplicial set given by Figure 1.3. Then a map $\hat{p} : K \star \Delta^0 \to \mathcal{C}$ is:

(a) An object (0-simplex) $\hat{V}$,

(b) A triple of morphism (1-simplices) $\hat{u}$, $\hat{v}$, $\hat{w}$,

(c) 2-simplices $\hat{\alpha}$ and $\hat{\beta}$ exhibiting the composites $\hat{v} \sim_{\hat{\alpha}} \hat{f} \circ \hat{u}$ and $\hat{v} \sim_{\hat{\beta}} \hat{g} \circ \hat{w}$.

$$\begin{tikzcd}
V \arrow[r, "u"] \arrow[d, "w"'] \arrow[dr, "v"] & \bullet \arrow[d, "f"] \arrow[dl, phantom, "\Downarrow\alpha", near start] \\
\bullet \arrow[r, "g"'] \arrow[ur, phantom, "\Uparrow\beta", near start] & \bullet
\end{tikzcd}$$

Figure 1.3: Picture of the simplicial set $\Delta^0 \star (\bullet \to \bullet \leftarrow \bullet) \cong \Delta^1 \times \Delta^1$

For $n = 1$, the lifting property of the limit implies that for any other set of simplices $\hat{V}', \hat{u}', \hat{v}', \hat{w}', \hat{\alpha}', \hat{\beta}'$ as in Figure 1.3, there is a realization of a cube with opposite faces of the form of precisely this same Figure 1.3, which with the front faces can be seen in Figure 1.4, inside $\mathcal{C}$, where every 2-simplex exhibits as composite its 1-faces, and there are 3-simplices relating all the possible composites as their 2-faces.

$$\begin{tikzcd}
V' \arrow[rr, "u'"] \arrow[dr] \arrow[drrr] \arrow[dd, "w'"'] \arrow[dddr] & & \bullet \arrow[dr] & \\
& V \arrow[rr, "u"'] \arrow[dd, "w"] \arrow[ddrr, "v"] & & \bullet \arrow[dd] \\
\bullet \arrow[dr] & & & \\
& \bullet \arrow[rr] & & \bullet
\end{tikzcd}$$

Figure 1.4: Lifting property of the limit $\hat{p} : \Delta^0 \star (\bullet \to \bullet \leftarrow \bullet) \to \mathcal{C}$, for $n = 1$.

The lifting property for higher n becomes harder to give as an explicit *geometric* description. Nevertheless, the case $n = 1$ already gives a lot of intuition. For example, in case we are only interested in 2-simplices, then the definition of the above limit is exactly the definition of a **homotopy pullback**, or **Pseudo pullback**, in the theory of bicategories [Lac06, Example 6.8].

We end our discussion on limits by considering (finite) products in quasicategories. Let I be a discrete simplicial set with finite objects and let $\mathcal{C}$ be a quasicategory, then a diagram $F : I \to \mathcal{C}$ is determined by the collection of objects $\{x_i := F(i)\}_{i \in I} \in \mathcal{C}$. Notice that $\Delta^0 \star I \cong \coprod_I \Delta^1 / \sim$, where $\sim$ identifies all the initial edges of the 1-simplices.

$$\begin{tikzcd}
& & & \bullet \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & & \\
\cdots & \bullet & \bullet & \bullet & \bullet & \bullet & \cdots
\end{tikzcd}$$

Thus a map $\Delta^0 \star I \to \mathcal{C}$ is determined by an objects $x \in \mathcal{C}$ and a collection of 1-simplices $\{p_i : x \to x_i\}_{i \in I}$. We say $(x, \{p_i : x \to x_i\}_{i \in I})$ is a **product** if the associated diagram $\Delta^0 \star I \to \mathcal{C}$ is a limit of $I \to \mathcal{C}$.

Proposition 1.3.9. Let I be finite set and let $\{x_i\}_{i \in I}$ be a collection of objects in a quasicategory $\mathcal{C}$. An object x together with a collection of maps $\{p_i : x \to x_i\}_{i \in I}$ is a product if and only if for every $y \in \mathcal{C}_0$ the induced morphism

$$\mathcal{C}(y, x) \to \prod_{i \in I} \mathcal{C}(y, x_i)$$

is an equivalence of Kan complexes.

Proof. See [Rez21, Section 61.10] or [Lur09a, Section 4.4.1] $\square$

Fibrations in Quasicategories

In the theory of presheaves in categories one usually work with fibrations of categories $C \to C'$, such fibration defines a family of fiber categories $\{C'_x\}_{x \in C'}$ indexed by the objects of C'. However these categories might not be *a priori* related, the notion required to make the fiber categories functorial with respect to morphism of C' is that of a coCartesian fibration, or Grothendieck fibration. This concept generalize to quasicategories via the following definitions.

Definition 1.3.10. Let $\mathcal{C}$ and $\mathcal{D}$ be quasicategories. A map of simplicial sets $\mathcal{C} \to \mathcal{D}$ is an **inner fibration** if for every $n \geq 2$ and $0 < i < n$ there exists a lift to each diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & \mathcal{C} \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathcal{D} \end{tikzcd}$$

One can check that the property of being a inner fibration is stable under taking pullbacks. Let $p : \mathcal{C} \to \mathcal{D}$ be an inner fibration, then upon taking taking the pullback over an object $x \in \mathcal{D}$

$$\begin{tikzcd} \mathcal{C}_x := \mathcal{C} \times_x \Delta^0 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta^0 \arrow[r, "x"'] & \mathcal{D} \end{tikzcd},$$

we can see that the simplicial set $\mathcal{C}_x$ satisfies the inner horn lifting property. Thus an inner fibration $\mathcal{C} \to \mathcal{D}$ parametrises a family of quasicategories indexed by the objects of $\mathcal{D}$. However this parametrisation is in general not functorial with respect to the morphism in $\mathcal{C}$. This is fixed in a similar way to the usual picture: by introducing a collection of morphisms with special lifting properties.

Definition 1.3.11. Let $\mathcal{C}$ and $\mathcal{C}'$ be quasicategories and $p : \mathcal{C} \to \mathcal{C}'$ a map of simplicial sets. A 1-simplex $f : x \mapsto y$ in $\mathcal{C}$ is a p**-cocartesian morphism** if there exists a lift for every diagram

$$\begin{tikzcd} \Delta^{(0,1)} \arrow[r] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\ & \Delta^n \arrow[r] \arrow[ur, dotted] & \mathcal{C}' \end{tikzcd}$$

Definition 1.3.12. A **coCartisian fibration** is an inner fibration $\mathcal{C} \to \mathcal{D}$ such that for 1-simplex $f : x \to y$ in $\mathcal{D}$ and every $\tilde{y}$ over y, there is a coCartesian edge $\tilde{f} : \tilde{x} \to \tilde{y}$ with the property $p(\tilde{f}) = f$.

In the categorical theory the coCartesian condition is exactly the lifting of edges with the required lifting properties for $1 \leq n \leq 2$. For quasicategories the story is similar: given a (inner) fibration $p : \mathcal{C} \to \mathcal{C}'$ a cocartesian $\hat{f}$ lift of a 1-simplex $f : x \mapsto y$ allows to construct a map of simplicial sets between the fibers $\hat{f}^! : \mathcal{C}_x \to \mathcal{C}_y$, see [Rez17, Section 64]. Since coCartesian morphisms satisfy some kind of *universal* lifting property, we could expect they are characterized as terminal objects in some quasicategory. It turns out this is indeed the case, see [Rez17, Proposition 64.12], thus in particular the space of cocartesian lifts is contractible.

Chapter 2

Examples of Quasicategories

In this chapter we will introduce the examples of quasicategories that will be used in later chapters. All of these example arise in the same way: given a functor $S : \Delta \to C$ and an object $c \in C_0$, then the Yoneda embedding defines a simplicial set

$$\Delta^{op} \xrightarrow{S} C^{op} \xrightarrow{\mathrm{Hom}_C(-,c)} \mathrm{Set}\,.$$

We have already seen examples of this construction: the nerve of a category and the simplicial chains functor. We saw that in both these cases the simplicial sets in consideration were quasicategories. Further examples of simplicial sets will arise from higher categories, we will be particularly interested in the nerves of bicategories, tricategories, and topological categories. Bicategories and tricategories are the first examples of higher categories: categories with n-morphism for $n \geq 2$, for which there exist concrete combinatorial models. Nerves of these higher categories are not always quasicategories, however under appropriate conditions involving invertibility of the higher n-morphism (for $n \geq 2$) they will define quasicategories (Theorem 2.2.13 and Theorem 2.3.9).

2.1 Topological Categories and the Homotopy Coherent Nerve

There is a *strict*, or *naive*, model for $(\infty, 1)$-categories, namely topological categories. In this section we introduce topological categories and describe a nerve construction that allows us to obtain a quasicategory from a topological category.

Definition 2.1.1. A **topological category** $\mathcal{D}$ is a category enriched over Top. Explicitly this means a set (or collection) of objects $\mathrm{ob}(\mathcal{D}) =: \mathcal{D}_0$ together with hom topological spaces $\mathcal{D}(x, y)$ for every pair of objects $x, y \in \mathcal{D}_0$, continuous compositions maps

$$\circ : \mathcal{D}(y, z) \times \mathcal{D}(x, y) \to \mathcal{D}(x, z),$$

and identity morphism $id_x \in \mathcal{D}(x, x)$, satisfying $(f \circ g) \circ h = f \circ (g \circ h)$ and $f \circ id_x = f = id_y \circ f$.

Similarly, considering enrichment in simplicial sets and Kan complexes, we define **simplicial categories** and **Kan categories**. As a consequence of the (Quillen) equivalence between Kan and Top, we regard the theory of topological categories and Kan categories as equivalent.

Example 2.1.2. The canonical example of a topological category is that of CW complexes. This is the category with objects CW spaces and for two CW spaces $X, Y \in \mathrm{CW}$ the homorphism spaces $\mathrm{CW}(X, Y)$ are given by continuous maps $X \to Y$ with the compact open topology.

Notice that in topological categories and simplicial categories the associative and units are strict, in contrast to the case of quasicategories where the composition is only associate up to higher morphism. It turns out that that the model of topological categories is too strict, this causes that many definitions one would like to have in a $(\infty, 1)$-category model to be cumbersome in this model. For example the definition of *weak* (co)limits requires *coherent (co)fibrant replacements* of all the spaces in the limit diagram (compare with the seamlessly easy definition of limits in the quasicategory model). Despite the mentioned disadvantages many examples of quasicategories can be constructed from this model. In this section we show how to assign to a simplicial category, and in particular to a topological category, a quasicategory via a nerve construction. Before describing the input functor $\Delta \to \mathrm{Cat}_{\mathrm{sSets}}$ for this nerve let us introduce some definitions.

Definition 2.1.3. Consider $[n] = (\to\to \cdots \to)$ as a graph. Define $P_{i,j}$ as the set of paths starting at i and finishing at j. For paths p and p', we say p' refines if p can be obtained from p' by joining some paths. Refinement endows $P_{i,j}$ with a poset structure, where $p \leq p'$ if p' refines p. Thus we can consider its nerve $\mathbb{P}_{i,j} := N(P_{i,j})$. Explicitly the set of n-simplices $(\mathbb{P}_{i,j})_n$ is the set of all possible paths from $i \to j$ of length at least n.

It is useful to understand $\mathbb{P}_{i,j}$ in a more geometric way. For this we compare $\mathbb{P}_{i,j}$ with $\square^{j-i-1} := (\Delta^1)^{\times(j-i-1)}$. For $j > i$ define a map of simplicial sets

$$c : \mathbb{P}_{i,j} \to \square^{j-i-1},$$

as follows: On 0-simplices the map sends a path $i_1 \to i_2 \cdots \to i_n$ to the binary sequence with 1s on every $i_k - i_{k-1}$ spot for every $1 \leq k \leq n$. On 1-simplices the maps sends a refinements $p \leq p'$ to the unique map between the binary sequences of p and p' which sends 0 to a 1 in all the spots witnessing p' as a refinement of p. On general n-simplices the map sends a chain of refinements $(p_1 \leq p_2, \ldots, p_{n-1} \leq p_n)$ to the product $(\alpha_1, \ldots, \alpha_{j-i-1})$ of n-simplices in Δ^1, where α_k is a degenerate simplex of and edge, if and only if $(k \to k+2) \to (k \to k+1 \to k+2)$ appears in the chain of refinements.

Example 2.1.4. As an example of the map c, let $j - i - 1 = 5$ so $j = i + 6$. Then via the isomorphism the 0-simplex $(0, 1, 1, 0, 1) \in (\square^5)_0 = \{0, 1\}^5$ is mapped to the path $(i \to i + 2 \to i + 3 \to i + 5 \to i + 6)$ and the 1-simplex $(0, 0, 1, 0, 0) \to (0, 1, 1, 0, 1)$ is mapped to the poset inclusion

$$(i \to i + 3 \to i + 6) \leq (i \to i + 2 \to i + 3 \to i + 5 \to i + 6).$$

Lemma 2.1.5. For $i > j$, $\mathbb{P}_{i,j}$ is empty. If $i = j$, then $\mathbb{P}_{i,j} = *$. If $j > i$, then there is an isomorphism of simplicial sets

$$\mathbb{P}_{i,j} \xrightarrow{\cong} \square^{j-i-1}.$$

Sketch of the proof: The two first assertions are clear. To prove the third we consider an inverse for c,

$$p : \square^{j-i-1} \to \mathbb{P}_{i,j}.$$

On 0-simplices it maps a binary sequence to the path stopping at every position with a 1 in the binary sequence. On 1-simplices, it maps a sequence of 1-simplices of Δ^1 to the unique refinement between the edges of the 1-simplices (which are considered as binary sequences). For general n-simplices the map sends $\alpha \in \square^{j-i-1}$ to the unique sequence of refinements $(p_1 \leq p_2, \ldots, p_{n-1} \leq p_n)$ such that $p_k \leq p_{k+1} = [\alpha_{k,k+1}]$. It is clear that this defines an inverse of 0-simplices and 1-simplices. We omit the details showing this defines an inverse to c for higher simplices. However this follows easily from the (not yet discussed) fact that the nerve of a category is a 2-coskeletal simplicial set (see Definition 2.4.3 and Theorem 2.4.6). $\square$

Definition 2.1.6. For every $n \in \mathbb{N}$, define the simplicial category $S[n]$ by the following data:

1. $S[n]$ has as objects the set $\{0, 1, \ldots, n\}$.
2. The hom simplicial sets of $S[n]$ are given by

$$S[n](i, j) = \mathbb{P}_{i,j},$$

where the composition operation is induced from the concatenation of paths $P_{j,k} \times P_{i,j} \to P_{i,k}$, and the units are induced by the constant paths in $P_{i,i}$.

Definition 2.1.7. For each increasing map $f : [n] \to [m]$ define a functor of simplicial categories $Sf : S[n] \to S[m]$ by the following data:

1. On objects it is given by f.
2. Define $\tilde{f} : P_{i,j} \to P_{f(i),f(j)}$ by mapping a path to its image under f. Since f is an increasing map, it follows that this map preserves the poset structure of on $P_{i,j}$. Therefor $\tilde{f}$ induces a map of simplicial sets $N\tilde{f} : \mathbb{P}_{i,j} \to \mathbb{P}_{f(i),f(j)}$. Define a map between hom simplicial sets by

$$N\tilde{f} : S[n](i, j) \to S[m](f(i), f(j)).$$

Definition 2.1.8. Define a functor $S : \Delta \to \mathrm{Cat}_{\mathrm{sSets}}$ by the following data

- On objects, that is for each $[n]$, the functor assigns the simplicial category $S[n]$.
- On morphisms, that is for each increasing map $f : [n] \to [m]$, the functor S assigns the functor between simplicial categories Sf.

Definition 2.1.9. The **homotopy coherent nerve** $N_h(\mathcal{D})$ of a simplicial category $\mathcal{D}$ is the simplicial set associated to the functor $S : \Delta \to \mathrm{Cat}_{\mathrm{sSets}}$ as in Definition 2.1.8. Precisely, it is uniquely defined by

$$[N(\mathcal{D})]_n = \mathrm{Cat}_{\mathrm{sSets}}(S[n]), \mathcal{D}).$$

We extend the definition to topological categories by considering the latter as Kan simplicial categories.

Proposition 2.1.10. Let $\mathcal{D}$ be a Kan category or a topological category. Then the homotopy coherent nerve of $N(C)$ is a quasicategory.

Proof. See [Lur09b, Proposition 1.1.5.10.] □

Example 2.1.11. We describe the structure of the simplicial categories $S[n]$ for low n, and as a consequence we also give a description of the low dimensional n-simplices of $N(\mathcal{D})$ for a topological category $\mathcal{D}$.

- **n=0**: The simplicial category $S[0]$ has just one object $*$ and the mapping space is $S[0](*, *) \cong \Delta^0$. Thus 0-simplices of $N(\mathcal{D})$ are given by an object x in $\mathcal{D}$ together with the identity morphism $id_x \in \mathcal{D}(x, x)$. In particular, 0-simplices of $N(\mathcal{D})$ are in bijection with $\mathcal{D}_0$.
- **n=1**: The simplicial category $S[1]$ has 2 objects $\{0, 1\}$ and the mapping spaces are given by

$$S[1](0, 0) \cong \Delta^0,$$
$$S[1](0, 1) \cong \square^0 \cong \Delta^0.$$

Thus a 1-simplex of $N(\mathcal{D})$ is given by a pair of objects $x, y \in \mathcal{D}_0$ and a point in $\mathcal{D}(x, y)$.

- **n=2**: The simplicial category $S[2]$ has 3 objects $\{0,1,2\}$ and the mapping spaces are given by
$$\begin{aligned} &S[2](0,0) \cong S[2](1,1) \cong S[2](2,2) \cong \Delta^0, \\ &S[2](0,1) \cong S[2](1,2) \cong \Delta^0, \\ &S[2](0,2) \cong \square^1 \cong \Delta^1. \end{aligned}$$
There is a unique map $S[2](1,2) \times S[2](0,1) \to S[2](0,2)$ given by the inclusion of the initial point $\{0\} \to \Delta^1$. Thus a 2-simplex of $N(\mathcal{D})$ is given by objects $x, y, w \in \mathcal{D}_0$, points $a_1 \in \mathcal{D}(x,y)$, $a_2 \in \mathcal{D}(y,z)$ and a path $\gamma \in \mathcal{D}(x,z)$ starting at $a_2 \circ a_1$ and ending at some $a_3 \in \mathcal{D}(x,z)$.

- **n=3**: The simplicial category $S[3]$ has 4 objects $\{0,1,2,3\}$ and the mapping spaces are given by
$$\begin{aligned} &S[3](0,0) \cong S[3](1,1) \cong S[3](2,2) \cong S[3](3,3) \cong \Delta^0, \\ &S[3](0,1) \cong S[3](1,2) \cong S[3](2,3) \cong \Delta^0, \\ &S[3](0,2) \cong S[3](1,3) \cong \Delta^1, \\ &S[3](0,3) \cong \square^2. \end{aligned}$$
There are maps
$$\begin{aligned} &S[3](0,1) \times S[2](1,3) \to S[3](0,3), \\ &S[3](0,2) \times S[2](2,3) \to S[3](0,3), \end{aligned}$$
given by the inclusion of the the intervals on opposite sides of the square $\square^2$. Thus a 3-simplex of $N(\mathcal{D})$ is given by objects $x, y, z, w \in \mathcal{D}_0$, points $a_1 \in \mathcal{D}(x,y)$, $a_2 \in \mathcal{D}(y,z)$, $a_3 \in \mathcal{D}(z,w)$, paths $\gamma_1 \in \mathcal{D}(x,z)$ and $\gamma_2 \in \mathcal{D}(y,w)$ and a map $\square^1 \to \mathcal{D}(x,z)$. The last map $\square^1 \to \mathcal{D}(x,z)$ is bounded by the paths $\gamma_1 \circ a_3, a_2 \circ \gamma_2$ so it may be regarded as homotopy between these paths.

Following the pattern we may intuitively consider n-simplices as describing homotopies of homotopies between the images of products inside $\mathcal{D}(0,n)$.

A category C can be realized as a Kan category where the Kan spaces between two objects x and y are given the by the discrete simplicial set $C(x,y)$. This defines a fully faithful functor
$$\mathrm{Cat} \to \mathrm{Cat}_{\mathrm{sSets}}.$$
The following lemma compares the possible nerves in view of this embedding. This will be useful in later chapters when we consider quasicategories whose mapping spaces are homotopic to discrete spaces.

Lemma 2.1.12. Let C be a category. Then the composition $S : \Delta \to \mathrm{Cat}_{\mathrm{sSets}} \to \mathrm{Sets}$ factors as
$$\begin{tikzcd} \Delta \arrow[r, "S"] \arrow[dr, dotted] & \mathrm{Cat}_{\mathrm{sSets}} \arrow[r, "\mathrm{Cat}_{\mathrm{sSets}}(-{,}C)"] & \mathrm{Set} \\ & \mathrm{Cat} \arrow[u, hook] \arrow[ur, "\mathrm{Cat}(-{,}C)"'] & \end{tikzcd}$$
In consequence, for a category C, considered as a discrete simplicial category, the homotopy coherent nerve agrees with the nerve of C as a category.

Proof. This follows from the following claim:
$$\mathrm{Cat}_{\mathrm{sSets}}(S[n], C) = \mathrm{Cat}(\Delta, C).$$

Indeed, elements on the left hand side are determined by maps between objects $F : (S[n])_0\Delta \to C_0$ and maps of simplicial sets $\mathbb{P}_{i,j} \to C(F(i), F(j))$ compatible with compositions and identities. Since $\mathbb{P}_{i,j}$ is connected (every pair of 0-simplices can be connected by a 1-simplex) and $C(F(i), F(j))$ is discrete we see

$$\mathrm{sSets}(N(P_{i,j}), C(F(i), F(j))) \cong C(F(i), F(j)),$$

and the data of a simplicial category functor is just the same as that of a functor $[n] \to C$. $\square$

As we have seen, much of the structure of a quasicategory can be obtained from its homotopy category and its mapping spaces. For the homotopic coherent nerve we have a nice characterization of the mapping spaces.

Proposition 2.1.13. Let $\mathcal{D}$ be topological category with connected mapping spaces and let $N(\mathcal{D})$ its homotopic coherent nerve. Let x, y be objects of $\mathcal{D}$ considered as objects of $N(\mathcal{D})$ (see example 2.1.11), then there is a homotopy equivalence

$$\mathcal{D}(x, y) \cong N(\mathcal{D})(x, y).$$

Proof. See [HK20]. $\square$

2.2 Bicategories and Nerves of Bicategories

Basic Definitions in Bicategory Theory

In this subsection we will the introduce the basic definitions to discuss bicategories. For further discussion on the theory we refer the reader to [Lei98] and [Gur13].

Definition 2.2.1. A **bicategory** $(\mathcal{B}, a, r, l)$ is given by the data of:

1. A set (or collection) of objects $ob(\mathcal{B}) := \mathcal{B}_0$ referred as 0**-cells**.

2. Categories $\mathcal{B}(A, B)$ for every pair $A, B \in \mathcal{B}_0$. The objects in these categories are referred as 1**-cells** and the morphisms as 2**-cells**.

3. Functors c_{ABC} and id_A

$$\begin{aligned}&c_{ABC} : \mathcal{B}(B, C) \times \mathcal{B}(A, B) \to \mathcal{B}(A, C)\\&\text{On objects} \quad (g, f) \mapsto g \circ f,\\&\text{On arrows} \quad (\beta, \alpha) \mapsto \beta * \alpha,\end{aligned}$$

   and $id_A : * \to \mathcal{B}(A, A)$ (thus id_A can be regarded as a 1-cell $id_A : A \to A$). The composition on objects $\circ$ is referred as **horizontal composition**, on morphisms $*$ as **vertical composition**, and id_A as the **identity** 1**-cell**. When there is no chance of confusion we will shorten $f \circ g$ by fg.

4. Natural isomorphisms a_{ABCD}, l_{AB} and r_{AB} referred as the **associator**, **left unitor** and **right unitor**

(a)

$$
\begin{tikzcd}
\mathcal{B}(C,D)\times\mathcal{B}(B,C)\times\mathcal{B}(A,B) \arrow[r, "id\times c_{ABC}"] \arrow[d, "c_{BCD}\times id"'] \arrow[dr, phantom, "\Nearrow a_{ABCD}"] & \mathcal{B}(C,D)\times\mathcal{B}(A,C) \arrow[d, "c_{ACD}"] \\
\mathcal{B}(B,D)\times\mathcal{B}(A,B) \arrow[r, "c_{ABC}"'] & \mathcal{B}(A,D)
\end{tikzcd}
$$

(b)

$$
\begin{tikzcd}
\mathcal{B}(A,B)\times * \arrow[d, "id\times id_A"'] \arrow[dr, "\sim"] \arrow[dr, phantom, shift right=2, "\Nearrow r_{AB}"'] & \\
\mathcal{B}(A,B)\times\mathcal{B}(A,A) \arrow[r, "c_{AAB}"'] & \mathcal{B}(A,B)
\end{tikzcd}
$$

(c)

$$
\begin{tikzcd}
* \times\mathcal{B}(A,B) \arrow[d, "id_B\times id"'] \arrow[dr, "\sim"] \arrow[dr, phantom, shift right=2, "\Nearrow l_{AB}"'] & \\
\mathcal{B}(A,B)\times\mathcal{B}(A,A) \arrow[r, "c_{ABB}"'] & \mathcal{B}(A,B)
\end{tikzcd}
$$

On the level of 1-cells these are 2-cells

$$a_{hgf} : h(gf) \xrightarrow{\sim} (hg)f, \quad r_f : f \circ id \xrightarrow{\sim} f, \quad l_f : id \circ f \xrightarrow{\sim} f.$$

The above data should satisfy the following diagrams:

1.

$$
\begin{tikzcd}
 & & ((kh)g)f & & \\
(kh)(gf) \arrow[urr, "a\circ(c\times id_g\times id_f)"] & & & & (k(hg))f \arrow[ull, "a*id_f"'] \\
 & k(h(gf)) \arrow[ul, "a\circ(id_k\times id_h\times c)"] \arrow[rr, "id_k*a"'] & & k((hg)f) \arrow[ur, "a\circ(id_k\times c\times id_f)"'] &
\end{tikzcd}
$$

2.

$$
\begin{tikzcd}
(g\circ id)\circ f \arrow[rr, "a"] \arrow[dr, "r*id"'] & & g\circ(id\circ g) \arrow[dl, "id*l"] \\
 & g\circ f &
\end{tikzcd}
$$

A bicategory is called a **2-category** if the natural transformations a, l and r are the identity natural transformations. In other words, a 2-category is a category enriched in categories. A bicategory is called **unitary**, or **normal**, if the natural transformations l and r (not necessarily a) are the identity natural transformations.

Example 2.2.2. The main example of a bicategory is the category Cat of all small categories. Here the 0-cells are all the small categories, the 1-cells are the functors between categories and

2-cells are given by natural transformations. Horizontal and vertical composition are given by composition of functors and natural transformations, respectively. The identity 1-cells and 2-cells are given by the identity functors and identity natural transformations. In fact Cat is actually a 2-category.

Example 2.2.3. Now we present an example of a bicategory that is is not a 2-category. The bicategory $\mathcal{B}$ has just one 0-cell, the 1-cells are given vector spaces, and 2-cells is given by linear maps. Horizontal composition is given by the tensor product, while the vertical composition is given by composition of linear maps. Since in general the tensor product of vector spaces is not associative, but only up to a linear map, the category is not strict. This example may be generalized to arbitrary monoidal categories and it is known as the **delooping bicategory** of a monoidal category.

Definition 2.2.4. A **(2,1)-bicategory** is a bicategory in which every 2-cell is invertible.

Just as in the case of categories we can define *higher functors* and *higher natural transformations* between bicategories. Following the creed of higher category theory these should satisfy the usual equalities of a functor or natural transformation up to 2-cells.

Definition 2.2.5. Let $\mathcal{B}$ and $\mathcal{B}'$ be bicategories. A **pseudofunctor** $(F,\phi):\mathcal{B}\to\mathcal{B}'$ is the data of:

1. A function $F:\mathcal{B}_0\to\mathcal{B}'_0$.
2. Functors $F_{AB}:\mathcal{B}(A,B)\to\mathcal{B}'(FA,FB)$.
3. Invertible natural transformations

   (a)
$$\begin{tikzcd} \mathcal{B}(B,C)\times\mathcal{B}(A,B) \arrow[r, "c"] \arrow[d, "F_{BC}\times F_{AB}"'] \arrow[dr, phantom, "\Nearrow \phi_{ABC}"] & \mathcal{B}(A,C) \arrow[d, "F_{AC}"] \\ \mathcal{B}'(FB,FD)\times\mathcal{B}'(FA,FB) \arrow[r, "c'"'] & \mathcal{B}'(FA,FC) \end{tikzcd}$$

   (b)
$$\begin{tikzcd} * \arrow[r, "id_A"] \arrow[d, equal] \arrow[dr, phantom, "\Nearrow \phi_A"] & \mathcal{B}(A,A) \arrow[d, "F_{AA}"] \\ * \arrow[r, "id'_{FA}"'] & \mathcal{B}'(FA,FA) \end{tikzcd}$$

   On the level of 1-cells these are 2-cells

$$\phi_{fg}:Fg\circ Ff\to F(g\circ f), \qquad \phi:id'\to Fid.$$

The above data should satisfy the coherence data determined by the following diagrams:

1.
$$\begin{tikzcd} (Fh\ Fg)Ff \arrow[r, "\phi_{h,g}*id"] \arrow[d, "a'"] & F(hg)Ff \arrow[r, "\phi_{hg,f}"] & F((hg)f) \arrow[d, "Fa"] \\ Fh(Fg\ Ff) \arrow[r, "id*\phi_{g,f}"] & Fh\ F(gf) \arrow[r, "\phi_{h,gf}"] & F(h(gf)) \end{tikzcd}$$

2.

$$\begin{tikzcd}
Ff \circ id'_{FA} \arrow[r, "\phi"] \arrow[d, "r'"] & Ff \circ F(id_A) \arrow[r, "id*\phi_{f,id_{F(A)}}"] & F(f \circ id_A) \arrow[d, "Fr"] \\
Ff \arrow[rr, equal] & & Ff
\end{tikzcd}$$

3.

$$\begin{tikzcd}
id'_{FB} \circ Ff \arrow[r, "\phi*Id"] \arrow[d, "l'"] & F(id_B) \circ Ff \arrow[r, "\phi_{id_B,f}"] & F(I_B \circ f) \arrow[d, "Fl"] \\
Ff \arrow[rr, equal] & & Ff
\end{tikzcd}$$

A pseudofunctor is called **strict** if the 2-cells $\phi_{f,g}$ and ϕ are the identity natural transformations. A pseudofunctor is called **unital**, or **normal**, if the 2-cells $\phi_{f,g}$ are all isomorphims and ϕ is the identity natural transformation.

It is easy to see that bicategories and pseudofunctors form a category which we will denote by Bicat. It is also easy to see that the different degrees of strictness considered in the definitions (stricness, unitarity) describe subcategories of Bicat. We will summarize, and introduce notation, for the different conditions that describe possible subcategories of Bicat.

	Bicategory data	Pseudofunctor data
Bicat	No conditions	No conditions
Bicat_U	$l = id$, $r = id$, 2-cells are isomorphisms	$\phi = id$, $\phi_{f,g}$ are isomorphisms.
$2\,\mathrm{Cat}_U$	$a = id$, $l = id$, $r = id$	$\phi = id$, $\phi_{f,g}$ are isomorphisms.
2Cat	$a = id$, $l = id$, $r = id$	$\phi = id$, $\phi_{f,g} = id$

Table 2.1: Some possible subcategories of Bicat and the conditions describing them.

For our purposes it is enough to consider only the category Bicat_U, thus the following definitions will be considered just in this framework. Nevertheless the reader should be able to recover the weaker versions of these definition, either by searching in the literature the names below without the adjective "unitary", or by adding 1-cells and coherence diagrams to handle non-unitarity.

Definition 2.2.6. Let (F, ϕ) and (G, ψ) be unital pseudofunctors between 2-categories. A **unital pseudonatural transformation** $\sigma : (F\phi) \Rightarrow (G, \psi)$ between is the data of:

1. 1-cells $FA \xrightarrow{\sigma_A} GA$, for each 0-cell A.
2. Invertible natural transformations σ_{AB}, for every pair of 0-cells A, B

$$\begin{tikzcd}
\mathcal{B}(A,B) \arrow[r, "F_{AB}"] \arrow[d, "G_{AB}"'] \arrow[dr, phantom, "\nearrow \sigma_{AB}"] & \mathcal{B}'(FA,FB) \arrow[d, "(\sigma_B)_*"] \\
\mathcal{B}'(GA,GB) \arrow[r, "(\sigma_A)^*"'] & \mathcal{B}'(FA,GB)
\end{tikzcd}$$

where $(\sigma_B)_*$, $(\sigma_A)^*$ are the functors induced from postcomposition and precompostion by σ_B respectively σ_A. On the level of 1-cells these are 2-cells

$$\sigma_f : Gf \circ \sigma_A \to \sigma_B \circ Ff$$

The above data should satisfy the following 2-cells diagram

$$\begin{tikzcd}
Gg \circ Gf \circ \sigma_A \arrow[r, "id_{Gg} * \sigma_f"] \arrow[d, "\psi_{f,g} * id_{\sigma_A}"] & Gg \circ \sigma_B \circ Ff \arrow[r, "\sigma_g * id_{Ff}"] & \sigma_C \circ Fg \circ Ff \arrow[d, "id_C * \phi_{g,f}"] \\
G(gf) \circ \sigma_A \arrow[rr, "\sigma_{gf}"] & & \sigma_C \circ F(gf)
\end{tikzcd} \tag{2.1}$$

Definition 2.2.7. Let $\mathcal{B}, \mathcal{D}$ be 2-categories, $F, G : \mathcal{B} \to \mathcal{D}$ unital pseudofunctors and $\sigma, \xi : F \Rightarrow G$ a unital pseudonatural transformation. A (unital) **modification** $\Gamma : \sigma \Rrightarrow \xi$ is the data of:

1. 2-cells Γ_A, for each 0-cell A,

such that the following 2-cells diagram holds

$$\begin{tikzcd}
Gf \circ \sigma_A \arrow[r, "id_{Gf} * \Gamma_A"] \arrow[d, "\sigma_f"] & Gf \circ \xi_A \arrow[d, "\xi_f"] \\
\sigma_B \circ Ff \arrow[r, "\Gamma_B * id_{Ff}"] & \xi_B \circ Ff
\end{tikzcd} \tag{2.2}$$

Pasting diagrams

In usual category theory properties between morphism are encoded in (1-dimensional) diagrams; similarly, in bicategory theory properties concerning 2-cells can be encoded in 2-dimensional diagrams, these are called **pasting diagrams**. A pasting diagram in a 2-category $\mathcal{B}$ is a labeled polygonal arrangement in the plane, such that the labeling satisfies: 1) vertices are labeled by 0-cells , 2) edges are labeled by 1-cells, and 3) faces are labeled by 2-cells. For example, consider the pasting diagram in Figure 2.1: $f : A \to B$, $g : B \to C$, etc., are 1-cells; meanwhile $\alpha : h \circ g \Rightarrow l \circ m$, $\gamma : l \circ k \Rightarrow j \circ n$, etc., are 2-cells.

$$\begin{tikzcd}
 & C \arrow[dr, "h"] & \\
B \arrow[ur, "g"] \arrow[dr, "m"'] \arrow[rr, phantom, "{}^1\Downarrow_\alpha"] & & D \\
 & E \arrow[ur, "l"'] \arrow[dd, "n"] & \\
A \arrow[uu, "f"] \arrow[dr, "i"'] \arrow[ur, phantom, "{}^3\Downarrow_\gamma"] & & G \arrow[uu, "k"'] \arrow[ul, phantom, "{}^2\Downarrow_\beta"] \\
 & F \arrow[ur, "j"'] &
\end{tikzcd}$$

Figure 2.1: Example of a pasting diagram.

Each pasting diagram defines a unique 2-cell from the outer edges, in the case of Figure 2.1 this is a 2-cell $h \circ g \circ f \Rightarrow k \circ j \circ i$. To define this 2-cell in term of composition, we should replace the step by step compositions of 1-cells by the 2-cells in the pasting diagram. For example in Figure 2.1 this series of steps are

$$(hg)f \xRightarrow{\alpha * id_f} (lm)f = l(mf) \xRightarrow{\beta * id_{mf}} (kjn)(mf) = (kj)(nmf) \xRightarrow{id_{kj} * \gamma} kji.$$

Thus the whole 2-cell defined by the pasting diagram is

$$(id_{kj} * \gamma) \circ (\beta * id_{mf}) \circ (\alpha * id_f).$$

When needed, we will write numbers on the two cells to guide the reader in order of vertical composition of the 2-cells in a pasting diagram, like in Figure 2.1.

Equalities of pasting diagrams have 3-dimensional intepretations: each pasting diagram can be though as a 2-dimensional polytope, the equality of two pasting diagrams can be interpreted as the existence of a 3-dimensional polytope constructed by gluing the pasting diagrams along the common edges. For example, in Figure 2.2 the equality of pasting diagrams can be interpreted as a cube.

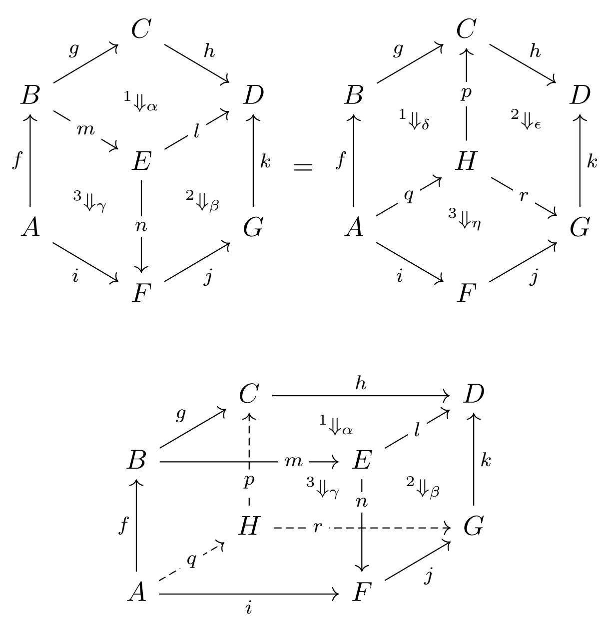


Figure 2.2: Equality of pasting diagrams interpreted as a 3-dimensional polytope. For clearness we just present the *front* 2-cells of the polytope.

Definition 2.2.8. An **adjunction** $(F \dashv F^\bullet, \eta, \epsilon)$ is a pair of 1-cells $F : X \to Y$, $F^\bullet : Y \to X$, and 2-cells $\eta : id_X \Rightarrow F^\bullet \circ F$, $\epsilon : F \circ F^\bullet \Rightarrow id_Y$ (unit and counit), such that the following equality of pasting diagrams hold

1.

$$
\begin{tikzcd}
 & Y \arrow[rr, equal] \arrow[dr, "F^\bullet"] \arrow[r, phantom, "\Uparrow\epsilon", near end] & & Y \\
X \arrow[ur, "F"] \arrow[rr, equal] \arrow[r, phantom, "\Uparrow\eta", shift left=3] & & X \arrow[ur, "F"] &
\end{tikzcd}
=
\begin{tikzcd}
 & Y \arrow[rr, equal] & & Y \\
X \arrow[ur, "F"] \arrow[rr, equal] \arrow[urrr, phantom, "="] & & X \arrow[ur, "F"] &
\end{tikzcd}
$$

2.

$$
\begin{tikzcd}
 & X \arrow[rr, equal] \arrow[dr, "F"] \arrow[r, phantom, "\Downarrow\eta", near end] & & X \\
Y \arrow[ur, "F^\bullet"] \arrow[rr, equal] \arrow[r, phantom, "\Downarrow\epsilon", shift left=3] & & Y \arrow[ur, "F^\bullet"] &
\end{tikzcd}
=
\begin{tikzcd}
 & X \arrow[rr, equal] & & X \\
Y \arrow[ur, "F"] \arrow[rr, equal] \arrow[urrr, phantom, "="] & & Y \arrow[ur, "F"] &
\end{tikzcd}
$$

An **adjoint equivalence** is an adjunction where the unit and counit are natural isomorphisms. Let F be a 1-cell, then a **adjunction data** is an adjoint equivalence $(F, F^\bullet, \eta_F, \epsilon_F)$.

We can use adjoint equivalences to invert the edges of a pasting diagram without changing the equality of a pasting diagram.

Lemma 2.2.9. Let $\gamma^\bullet$ and $\beta^\bullet$ be the 2-cells defined by the following pasting diagrams

$$
\gamma^\bullet =
\begin{tikzcd}
B \arrow[r, "m"] & E \arrow[d, "n"] & \\
A \arrow[u, "f"] \arrow[r, "i"'] \arrow[ur, phantom, "{}^1\Downarrow_\gamma"] & F \arrow[r, equal] \arrow[ur, phantom, "{}^2\Uparrow_\epsilon"] & F \arrow[ul, "n^\bullet"']
\end{tikzcd}
, \qquad
\beta^\bullet =
\begin{tikzcd}
E \arrow[r, equal] \arrow[dr, phantom, "{}^1\Downarrow_\eta"] & E \arrow[r, "l"] \arrow[d, "n"] \arrow[dr, phantom, "{}^2\Downarrow_\beta"] & D \\
 & F \arrow[ul, "n^\bullet"] \arrow[r, "j"'] & G \arrow[u, "k"']
\end{tikzcd}
.
$$

Then there is an equality of pasting diagrams

$$
\begin{tikzcd}
B \arrow[r, "m"] & E \arrow[r, "l"] \arrow[d, "n"] & D \\
A \arrow[u, "f"] \arrow[r, "i"'] \arrow[ur, phantom, "{}^2\Downarrow_\gamma"] & F \arrow[r, "j"'] \arrow[ur, phantom, "{}^1\Downarrow_\beta"] & G \arrow[u, "k"']
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, "m"] & E \arrow[r, "l"] & D \\
A \arrow[u, "f"] \arrow[r, "i"'] \arrow[ur, phantom, "{}^1\Downarrow_{\gamma^\bullet}"] & F \arrow[u, "n^\bullet"] \arrow[r, "j"'] \arrow[ur, phantom, "{}^2\Downarrow_{\beta^\bullet}"] & G \arrow[u, "k"']
\end{tikzcd}
$$

Proof. The statement is clear from the definitions of $\gamma^\bullet$, $\beta^\bullet$ and the condition on units and counits of adjoint equivalences. ☐

In an equality of pasting diagrams in which all the edges are adjoint equivalences an inductive use of Lemma 2.2.9 allows to check equality of pasting diagrams by examining the (unoriented) polytope it defines (as long as the boundaries have the same orientation). See [KV94a] and [Gur13] for a in depth discussion on this. This will be constantly used in the proofs the main theorems in this work Theorems 0.0.3, 0.0.4, 0.0.4 and 0.0.6 in the introduction.

Remark 2.2.10. We restricted the definition of pasting diagrams to 2-categories because in this case the strictness of 2-categories ensures that pasting diagrams define a unique 2-cell between its edges. One may also define pasting diagrams in bicategories, however in this case one should be more careful about why the diagrams are well defined. Lets sketch the problem: In a bicategory different bracketings in a composition of 1-cells yields a different result (related by an associator). For example from the diagram in Figure 2.1 is not clear whether the morphism maps $h(gf) \Rightarrow i(jk)$, or $h(gf) \Rightarrow (ij)k$, or maybe other possibility of bracketings. Moreover it may be required to use different bracketings on 1-cells to make sense of the pasting diagram. For example $\beta * id_{mf}$ and $id_{kj} * \gamma$ have different source and targets, thus we require to use several associators to make sense of a pasting diagram. MacLane coherence theorem [Gur13] ensures that there are canonical coherence isomorphisms between the bracketed expressions. Therefore, upon choosing a bracketing on the edges. the pasting diagram is well defined (although the interpretation as a 2-cell requires keeping track of the additional associators).

Nerve of Bicategories

Definition 2.2.11. Let $\mathcal{B}$ be a bicategory. The **nerve of a bicategory** $N_2\mathcal{B}$ is the nerve associated to the functor

$$\Delta \to \mathrm{Cat} \to \mathrm{Bicat}_U,$$

given by considering the categories $[n]$ as 2-categories with identity 2-cells. Explicitly the n-simplices of $[N_2(\mathcal{B})]_n$ are given by

1. A collection of objects $\{X_i\}_{0\leq i\leq n}$.
2. A collection of 1-cells $\{f_{j,i} : X_i \to X_j\}_{0\leq i\leq j\leq n}$ in the bicategory $\mathcal{B}$.
3. A collection of 2-cells $\{\mu_{k,j,i} : f_{k,j} \circ f_{j,i} \Rightarrow f_{k,i}\}_{0\leq i\leq j\leq k\leq n}$.

Subject to the conditions:

1. $f_{i,i} = id_{X_i}$, for $0 \leq i \leq n$.
2. $\mu_{i,i,j} = id_{f_{j,i}}$, for $0 \leq i \leq j \leq n$.
3. $\mu_{i,j,j} = id_{f_{i,j}}$, for $0 \leq i \leq j \leq n$.
4. For $0 \leq i < j < k < l \leq n$, the following 2-cell diagram holds in the category $\mathcal{B}(X_i, X_l)$

$$\begin{tikzcd}
 & (f_{l,k}f_{k,j})f_{j,i} \arrow[r, Rightarrow, "a"] \arrow[dl, Rightarrow, "\mu_{l,k,j}*id_{f_{i,j}}"'] & f_{l,k}(f_{k,j}f_{j,i}) \arrow[dr, Rightarrow, "id_{f_{l,k}}*\mu_{k,j,i}"] & \\
f_{l,k}f_{k,i} \arrow[drr, Rightarrow, "\mu_{l,j,i}"'] & & & f_{l,j}f_{j,i} \arrow[dl, Rightarrow, "\mu_{l,k,i}"] \\
 & & f_{l,i} &
\end{tikzcd}$$

The simplicial operators are induced by precomposition by the simplicial operator in Δ, i.e. face maps are given by composition or projections; while the degeneracy maps are given by repeating elements and adding identities.

These collections are just the images of $[n]$, with n being the number of indices of the collection. The diagrams appearing in the definition are just the ones from the definition of unitary pseudofunctor, see Definition 2.2.5.

Remark 2.2.12. It is useful to consider a geometric picture for the first simplices of the nerve of a bicategory:

1. For $n = 0$: There is just one object X_1, and the higher cells must be the identity cells. The 0-simplices are determined by just one object X_1, and can be visualized as points.

2. For $n = 1$: There are 2 objects X_1, X_2, one non trivial 1-Cell $f_{i,j}$, and 2-cells must be the identity cells. Thus 1-simplices are determined by a 1-cell $f_{12} : X_1 \to X_2$, and can be visualised as lines.

3. For $n = 2$: There are 0-cells X_1, X_2, X_3; 1-cells f_{12}, f_{23}, f_{13}, and a 2-cell $\mu_{123} : f_{13} \circ f_{12} \Rightarrow f_{13}$. Thus 2-simplices can be visualized as diagrams

$$\begin{tikzcd}
& x \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow a"] & \\
y \arrow[ur, "f"] \arrow[rr, "h"'] & {} & z
\end{tikzcd}$$

   Notice that the 2-cell maps *out* from the composition.

4. For $n = 3$: Following the same pattern above we can visualize 3-simplices as the faces of a tetrahedra with an orientation given by the rule that a 2-cell points *out* of the composition of 1-cells.

5. For $n \geq 4$: The 4-simplices are given by all the possible ways to paste the tetrahedra described by triples of objects in $\{X_i\}_{0 \leq i \leq n}$, where the faces of the different tetrahedra agree.

Theorem 2.2.13. Let $\mathcal{B}$ be a $(2,1)$-category then $N_2(\mathcal{B})$ is a quasicategory.

Proof. The proof can be found in [Dus02, Section 6]. □

The main use of Theorem 2.2.13 is that it immediately implies theorem 0.0.1 stated in the introduction: the nerve of the (2,1)-category $\mathrm{Cat}_{(2,1)}$ of categories, functors and invertible natural transformation, denoted $\mathbb{C}\mathrm{at} := N_2(\mathrm{Cat}_{(2,1)})$, is a quasicategory.

2.3 Tricategories and Nerves of Tricategories

We can go one step higher and discuss tricategories. The definition and construction of the nerve of tricategories is completely analogous to that of bicategories. For simplicity we will not deal with the general theory of tricategories. Instead we will focus on the explicit example Bicat: the tricategory of bicategories, pseudofunctors, pseudonatural transformations and modifications. Nevertheless, many constructions presented in this section generalise without much effort to general tricategories. For a reference the general theory we refer the reader to [Gur13]. We will provide (incomplete) definitions which omit the (large) coherence diagrams present in the theory of tricategories.

Definition 2.3.1. A **tricategory** $(\mathcal{T}, a, r, l, \pi, \mu, \lambda, \mu)$ is the data of:

1. A set (or collection) of objects $ob(\mathcal{T}) := \mathcal{T}_0$ referred as 0**-cells**.

2. Bicategories $\mathcal{T}(A, B)$ for every pair $A, B \in \mathcal{T}_0$. The 0-cells in $\mathcal{T}(A, B)$ are referred as 1**-cells in** $\mathcal{T}$, 1-cells in $\mathcal{T}(A, B)$ as 2**-cells in** $\mathcal{T}$, and 2-cells in $\mathcal{T}(A, B)$ as 3**-cells in** $\mathcal{T}$.

3. Pseudofunctors $c_{ABC} : \mathcal{T}(B, C) \times \mathcal{T}(A, B) \to \mathcal{T}(A, C)$ and $id_A : * \to \mathcal{T}(A, A)$. Notice id_A can be regarded as a pair of a 1-cell $id_A : A \to A$ and a 2-cell $\tilde{id}_A : id_A \to id_A$.

4. Adjoint equivalences a_{ABCD}, l_{AB} and r_{AB} (in the bicategory of pseudonatural transformations and modifications) referred as the **associator**, **left unitor** and **right unitor**

(a)

$$\begin{tikzcd}
\mathcal{T}(C,D)\times\mathcal{T}(B,C)\times\mathcal{T}(A,B) \arrow[r, "id\times c_{ABC}"] \arrow[d, "c_{BCD}\times id"'] & \mathcal{T}(C,D)\times\mathcal{T}(A,C) \arrow[d, "c_{ACD}"] \arrow[dl, phantom, "\Nearrow a_{ABCD}"] \\
\mathcal{T}(B,D)\times\mathcal{T}(A,B) \arrow[r, "c_{ABC}"'] & \mathcal{T}(A,D)
\end{tikzcd}$$

(b)

$$\begin{tikzcd}
\mathcal{T}(A,B)\times * \arrow[d, "id\times id_A"'] \arrow[dr, "\sim"] & \\
\mathcal{T}(A,B)\times\mathcal{T}(A,A) \arrow[r, "c_{AAB}"'] \arrow[ur, phantom, "\Nearrow r_{AB}"] & \mathcal{T}(A,B)
\end{tikzcd}$$

(c)

$$\begin{tikzcd}
* \times\mathcal{T}(A,B) \arrow[d, "id_B\times id"'] \arrow[dr, "\sim"] & \\
\mathcal{T}(A,B)\times\mathcal{T}(A,A) \arrow[r, "c_{ABB}"'] \arrow[ur, phantom, "\Nearrow l_{AB}"] & \mathcal{T}(A,B)
\end{tikzcd}$$

5. Invertible modifications π_{ABCD}, μ_{ABC}, λ_{ABC} and ρ_{ABC} referred as **pentagonator** and **higher unitors**

(a)

$$\begin{tikzcd}
& \underline{\mathcal{T}}^4 \arrow[rr, "c\times1\times1"] \arrow[dl, "1\times1\times c"'] \arrow[dr, dashed, "1\times c\times1"] & & \underline{\mathcal{T}}^3 \arrow[dr, "c\times1"] \arrow[d, phantom, "\Downarrow_{a\times1}"] & & & & \underline{\mathcal{T}}^4 \arrow[rr, "c\times1\times1"] \arrow[dl, "1\times1\times c"'] \arrow[d, phantom, "="] & & \underline{\mathcal{T}}^3 \arrow[dl, dashed, "1\times c"'] \arrow[dr, "c\times1"] & \\
\underline{\mathcal{T}}^3 \arrow[dr, "1\times c"'] \arrow[r, phantom, "\Leftarrow_{1\times a}"] & & \underline{\mathcal{T}}^3 \arrow[rr, "c\times1"] \arrow[dl, dashed, "1\times c"'] \arrow[dr, phantom, "\Downarrow_{a\times1}"] & & \underline{\mathcal{T}}^2 \arrow[dl, "c"] \arrow[rr, phantom, "\Rrightarrow_\pi"] & & \underline{\mathcal{T}}^3 \arrow[rr, "c\times1"] \arrow[dr, "1\times c"'] & & \underline{\mathcal{T}}^2 \arrow[dr, "c"] \arrow[dl, phantom, "\Downarrow_a"] \arrow[rr, phantom, "\Leftarrow_a"] & & \underline{\mathcal{T}}^2 \arrow[dl, "c"] \\
& \underline{\mathcal{T}}^2 \arrow[rr, "c"'] & & \underline{\mathcal{T}} & & & & \underline{\mathcal{T}}^2 \arrow[rr, "c"'] & & \underline{\mathcal{T}} &
\end{tikzcd}$$

(b)

$$\begin{tikzcd}
\underline{\mathcal{T}}^2 \arrow[drr, bend left] \arrow[dr, dashed, "r^\bullet\times1"] \arrow[ddr, bend right] & & & & \underline{\mathcal{T}}^2 \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "1"'] & & \\
\Leftarrow_{1\times l} & \underline{\mathcal{T}}^3 \arrow[r, "c\times1"] \arrow[d, dashed, "1\times c"] \arrow[u, phantom, "\Downarrow_a"] \arrow[dr, phantom, "\Downarrow_a"] & \underline{\mathcal{T}}^2 \arrow[d, "c"] \arrow[rr, phantom, "\Rrightarrow_\mu"] & & & \Downarrow_1 & \underline{\mathcal{T}}^2 \arrow[d, "c"] \\
& \underline{\mathcal{T}}^2 \arrow[r, "c"'] & \underline{\mathcal{T}} & & & \underline{\mathcal{T}}^2 \arrow[r, "c"'] & \underline{\mathcal{T}}
\end{tikzcd}$$

(c)

$$
\begin{tikzcd}
& \underline{\mathcal{T}}^3 \arrow[dr, "c\times 1"] \arrow[d, phantom, "\Downarrow_{l\times 1}"] & \\
\underline{\mathcal{T}}^2 \arrow[ur, "id\times 1\times 1"] \arrow[rr, "1"] \arrow[d, "c"'] & {} & \underline{\mathcal{T}}^2 \arrow[d, "c"] \\
\underline{\mathcal{T}} \arrow[rr, "1"'] & {} \arrow[u, phantom, "="] & \underline{\mathcal{T}}
\end{tikzcd}
\Rrightarrow_{\lambda}
\begin{tikzcd}
& \underline{\mathcal{T}}^3 \arrow[dr, "c\times 1"] \arrow[dd, "1\times c"] & \\
\underline{\mathcal{T}}^2 \arrow[ur, "id\times 1\times 1"] \arrow[dd, "c"'] \arrow[r, phantom, "="] & {} \arrow[r, phantom, "\Downarrow_a"] & \underline{\mathcal{T}}^2 \arrow[dd, "c"] \\
& \underline{\mathcal{T}}^2 \arrow[dr, "c"] \arrow[d, phantom, "\Downarrow_l"] & \\
\underline{\mathcal{T}} \arrow[ur, "id\times 1"] \arrow[rr, "1"'] & {} & \underline{\mathcal{T}}
\end{tikzcd}
$$

(d)

$$
\begin{tikzcd}
& \underline{\mathcal{T}} \arrow[r, "1"] & \underline{\mathcal{T}} & \\
& \underline{\mathcal{T}}^2 \arrow[u, "c"] \arrow[r, "1"] \arrow[ur, phantom, "="] \arrow[d, "1\times 1\times id"'] & \underline{\mathcal{T}}^2 \arrow[u, "c"'] & \\
& \underline{\mathcal{T}}^3 \arrow[ur, "1\times c"'] \arrow[u, phantom, "\Downarrow_{1\times r^\bullet}"] & &
\end{tikzcd}
\Rrightarrow_{\rho}
\begin{tikzcd}
\underline{\mathcal{T}} \arrow[rr, "1"] \arrow[dr, "1\times id"'] & \Downarrow_{r^\bullet} & \underline{\mathcal{T}} \\
& \underline{\mathcal{T}}^2 \arrow[ur, "c"'] \arrow[l, phantom, "="] \arrow[r, phantom, "\Rrightarrow_a"] & {} \\
\underline{\mathcal{T}}^2 \arrow[uu, "c"] \arrow[dr, "1\times 1\times id"'] & & \underline{\mathcal{T}}^2 \arrow[uu, "c"'] \\
& \underline{\mathcal{T}}^3 \arrow[uu, "c\times 1"] \arrow[ur, "1\times c"'] &
\end{tikzcd}
$$

In the above diagrams, to ease notation, we use $\underline{\mathcal{T}}$ to denote a hom bicategory $\mathcal{T}(-,-)$, and 1 denotes the identity pseudofunctor.

The above data should satisfy the coherence diagrams as found in [Gur13, Definition 4.1].

Example 2.3.2. The collection of bicategories is a tricategory Bicat. For $\mathcal{B}$ and $\mathcal{B}'$ bicategories the mappping bicategory Bicat$(\mathcal{B},\mathcal{B}')$ has: pseudofunctors as 0-cells, pseudonatural tranformations as 1-cells and modifications as 2-cells. Thus in Bicat: 0-cells are bicategories, 1-cells are pseudofunctors, 2-cells are pseudonatural transformations, and 3-cells are modifications. Checking that Bicat is a tricategory is a straightforward consequence of the definitions (although it requires extensive checking). For details see [Gur13, Section 5.1]

Definition 2.3.3. We say a tricategory $\mathcal{T}$ is a **(3,1)-tricategory** if:

1. for any 2-cells $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow k$ in $\mathcal{T}(A,B)$, the functors

$$
\begin{aligned}
-\circ\alpha &: [\mathcal{T}(A,B)](g,h) \to [\mathcal{T}(A,B)](f,h),\\
\beta\circ - &: [\mathcal{T}(A,B)](f,g) \to [\mathcal{T}(A,B)](f,k),
\end{aligned}
$$

are equivalences of categories.

2. All 3-cells are invertible.

Proposition 2.3.4. Let $\mathcal{T}$ be a tricategory in which every 3-cell is invertible and every 2-cell α is part of an adjoint equivalence $(\alpha,\alpha^\bullet,\eta,\epsilon)$, then $\mathcal{T}$ is a $(3,1)$-tricategory.

Proof. An adjoint equivalence induces an equivalence of categories. Indeed, the functors $-\circ\alpha$ and $\alpha\circ -$ have inverses determined by $-\circ\alpha^\bullet$ and $\alpha^\bullet\circ -$. Notice that we are considering the 2-cell α in $\mathcal{T}$ as a 1-cell in some hom bicategory $\mathcal{T}(-,-)$. $\square$

One may consider different levels of strictness to define different subtricategories inside Bicat. For example we might restrict to just unitary data, strict data, or consider 2-categories but allow weak higher morphisms between them. In table 2.2 we summarize some of the possible subtricategories that will play important roles in the future.

	0-cells	1-cells	2-cells	3-cells
Bicat	Bicategories	Pseudofunctors	Pseudonatural transformations	Modifications
$\mathrm{Bicat}_{(3,1)}$	Bicategories	Pseudofunctors	Adjoint equivalences	Invertible Modifications
Gray	2-categories	2-functors	Pseudonatural transformations	Modifications
$\mathrm{Gray}_{(3,1)}$	2-categories	2-functors	Adjoint equivalences	Invertible modifications
2Cat	2-categories	2-functors	Strict natural transformations	Strict modifications

Table 2.2: Some possible subtricategories of Bicat and the conditions describing their cells.

Remark 2.3.5. Contrary to the bicategorical picture, not every tricategory is equivalent to a strict one. For example Bicat is not equivalent to neither to 2 Cat nor to Gray [Lac06]. Nevertheless, Bicat is triequivalent to a subtricategory Gray$'$ of Gray. For precise statements and proofs we refer the reader to [Gur13, Sections 8-10].

Pastings diagrams for Tricategories

In the previous section we have used some pasting diagrams to describe properties between pseudofunctors, pseudonatural transformation and modification. As in the case of pasting diagrams in bicategories to make sense of these diagrams some discussion is needed. Lets consider how the picture is different from the coherence for diagrams in bicategories. Consider the following data: pseudofunctors $H_i : \mathcal{B}_1 \to \mathcal{B}_2$, $G_i : \mathcal{B}_2 \to \mathcal{B}_3$, and $F_i : \mathcal{B}_3 \to \mathcal{B}_4$, for $i = 1, 2$; natural transformations $\theta_1 : H_1 \Rightarrow H_2$, $\theta_2 : G_2 \Rightarrow G_1$, $\theta_3 : F_2 \Rightarrow F_1$, $\alpha_1 : G_1 \circ H_1 \Rightarrow G_2 \circ H_2$, and $\alpha_2 : F_1 \circ G_1 \Rightarrow F_2 \circ G_1$; and modifications $\omega_1 : G_1\theta_1 \Rrightarrow (\theta_2 H_1) \circ \alpha_1$ and $\omega_2 : F_1\theta_2 \Rrightarrow (\theta_3 G_1) \circ \alpha_2$. With this data we can construct the pasting diagram in Figure 2.3.

$$\begin{tikzcd}
F_1G_1H_1 \arrow[r, "F_1G_1\theta_1"] \arrow[d, "F_1\alpha_1"'] \arrow[dr, phantom, "\Downarrow_{F_1\omega_1}"] & F_1G_2H_2 \\
F_1G_1H_2 \arrow[ur, "F_1\theta_2H_2"] \arrow[r, "\alpha_2H_2"'] & F_2G_1H_2 \arrow[u, "\theta_3G_1H_2"'] \arrow[ul, phantom, "\Downarrow_{\omega_2H_2}"]
\end{tikzcd}$$

Figure 2.3: Example of a pasting diagram between bicategories

Lets discuss whether this makes sense. The problem lies in the fact that $F_1((\theta_2H_2) \circ \alpha_1)$ and $(F_1\theta_2H_2) \circ F_1\alpha_1$ have different sources and targets, since in general the pseudofunctor F_1

might not be strict. However, in contrast to our discussion for bicategories, these unmatchings of sources and targets cannot be solved using just associators in the ambient bicategory. To solve the issue we need a theorem that strictifies both the bicategories as well as pseudofunctors between them.

In general a pseudofunctor is not equivalent to a strict pseudofunctor. However for a certain class of bicategories, known as **cofibrant**, it is the case that every pseudofunctor is equivalent to a strict functor. The name comes from the fact that they are the cofibrant objects in a model structure on the category of bicategories and pseudofunctors. There is a strictifying cofibrant replacement functor *st* which simultaneously strictify the bicategories and pseudofunctors in question, and upon choosing a bracketing on the edges such strictification is unique. Then it is clear that the pasting diagram makes sense and determines a well defined 2-cell. For a more in depth discussion we refer the reader to [SP09] section 2.2 and the references therein. It is worth mentioning that in the case that we are dealing with just 2-categories and 2-functors, that will be our main setup in later sections, then there is no need to introduce all the previous discussion and the pasting diagrams make sense *on the nose*.

Nerves of Tricategories

We now introduce the nerve of a tricategory. The construction, description and general discussion of nerves of tricategories is completely parallel to that of the nerves of bicategories.

Definition 2.3.6. Let $\mathcal{T}$ and $\mathcal{T}'$ be tricategories. A **unitary trihomomorphism** $(F, \phi, \omega, \gamma, \delta) : \mathcal{T} \to \mathcal{T}'$ is the data of:

1. A function $F : \mathcal{T}_0 \to \mathcal{T}'_0$.
2. Unitary pseudofunctors $F : \mathcal{T}(A, B) \to \mathcal{T}(FA, FA)$.
3. Adjoint equivalences

$$\begin{tikzcd}
\mathcal{T}(B,C)\times\mathcal{T}(A,B) \arrow[r, "c"] \arrow[d, "F_{BC}\times F_{AB}"'] \arrow[dr, phantom, "\Rightarrow \phi_{ABC}"] & \mathcal{T}(A,C) \arrow[d, "F_{AC}"] \\
\mathcal{T}'(FB,FD)\times\mathcal{T}'(FA,FB) \arrow[r, "c'"'] & \mathcal{T}'(FA,FC)
\end{tikzcd}$$

4. Modifications, (ω, γ, δ)

   (a)

$$\begin{tikzcd}
 & \underline{\mathcal{T}}^3 \arrow[rr, "F\times F\times F"] \arrow[dl, "1\times c"'] \arrow[dr, "c\times 1"] & \arrow[d, phantom, "\Downarrow_{\phi\times 1}"] & \underline{\mathcal{T}}'^3 \arrow[dr, "c'\times 1"] & & & & \underline{\mathcal{T}}^3 \arrow[rr, "F\times F\times F"] \arrow[dl, "1\times c"'] \arrow[d, phantom, "\Downarrow_{1\times\phi}"] & & \underline{\mathcal{T}}'^3 \arrow[dl, "1\times c'"'] \arrow[dr, "c'\times 1"] & \\
\underline{\mathcal{T}}^2 \arrow[dr, "c"'] \arrow[r, phantom, "\Leftarrow_a"] & {} & \underline{\mathcal{T}}^2 \arrow[rr, "F\times F"] \arrow[dl, "c"] & \arrow[d, phantom, "\Downarrow_\phi"] & \underline{\mathcal{T}}'^2 \arrow[dl, "c'"] & \Rrightarrow_\omega & \underline{\mathcal{T}}^2 \arrow[rr, "F\times F"] \arrow[dr, "c"'] & \arrow[d, phantom, "\Downarrow_\phi"] & \underline{\mathcal{T}}'^2 \arrow[dr, "c'"] & \Leftarrow_{a'} & \underline{\mathcal{T}}'^2 \arrow[dl, "c'"] \\
 & \underline{\mathcal{T}} \arrow[rr, "F"'] & & \underline{\mathcal{T}}' & & & & \underline{\mathcal{T}} \arrow[rr, "F"'] & & \underline{\mathcal{T}}' &
\end{tikzcd}$$

(b)

$$
\begin{tikzcd}
 & \underline{\mathcal{T}}'^2 \arrow[dr, "c\times 1"] & & & & \underline{\mathcal{T}}'^2 \arrow[dr, "c'"] \arrow[d, phantom, "\Downarrow_{l'}"] & \\
\underline{\mathcal{T}}' \arrow[ur, "id\times 1"] \arrow[dr, phantom, "="] & & \underline{\mathcal{T}}' & & \underline{\mathcal{T}}'2 \arrow[ur, "id\times 1"] \arrow[rr, "1"] \arrow[ddrr, phantom, "="] & {} & \underline{\mathcal{T}}' \\
 & \underline{\mathcal{T}}^2 \arrow[uu, "F\times F"] \arrow[dr, "c"] \arrow[ur, phantom, "\Rrightarrow_{\phi}"] \arrow[d, phantom, "\Downarrow_l"] & & \Rrightarrow_{\gamma} & & & \\
\underline{\mathcal{T}} \arrow[uu, "F"] \arrow[ur, "id\times 1"] \arrow[rr, "1"'] & {} & \underline{\mathcal{T}} \arrow[uu, "F"'] & & \underline{\mathcal{T}} \arrow[uu, "F"] \arrow[rr, "1"'] & & \underline{\mathcal{T}} \arrow[uu, "F"']
\end{tikzcd}
$$

(c)

$$
\begin{tikzcd}
\underline{\mathcal{T}}' \arrow[rr, "1"] \arrow[drr, phantom, "="] & & \underline{\mathcal{T}}' & & \underline{\mathcal{T}}' \arrow[rr, "1"] \arrow[dr, "1\times id'"] & \Downarrow_{r'^{\bullet}} & \underline{\mathcal{T}} \\
 & & & & = & \underline{\mathcal{T}}' \arrow[ur, "c"] & \Rrightarrow_{\phi} \\
\underline{\mathcal{T}} \arrow[uu, "F"] \arrow[rr, "1"] \arrow[dr, "1\times id"'] & \Downarrow_{r^{\bullet}} & \underline{\mathcal{T}} \arrow[uu, "F"'] & \Rrightarrow_{\delta} & \underline{\mathcal{T}} \arrow[uu, "F"] \arrow[dr, "1\times id"'] & & \underline{\mathcal{T}} \arrow[uu, "F"'] \\
 & \underline{\mathcal{T}}^2 \arrow[ur, "c"'] & & & & \underline{\mathcal{T}}^2 \arrow[uu, "F\times F"] \arrow[ur, "c"'] &
\end{tikzcd}
$$

In the above diagrams we use $\underline{\mathcal{T}}$ to denote a hom bicategory $\mathcal{T}(-,-)$, and 1 denotes the identity pseudofunctor.

The data should satisfy:

1. $F(id_A) = id_{FA}$ and the following diagram commutes

$$
\begin{tikzcd}
* \arrow[rr, "id_{FA}"] \arrow[dr, "id_A"'] & & \underline{\mathcal{T}}'(FA, FA) \\
 & \underline{\mathcal{T}}(A, A) \arrow[ur, "F"'] &
\end{tikzcd}
$$

2. For any 1-cell $f : A \to B$, it holds

$$\phi_{id_A, f} = l_{Ff} \quad \text{and} \quad \phi_{f, Id_A} = r^{\bullet}_{Ff}.$$

3. The diagrams in [Gur13, Definition 4.10] holds. For our purposes we will just write the constraint diagrams in the case the domain tricategory $\mathcal{T}$ is strict:

(a)

$$
\begin{tikzcd}[column sep=small]
& F(fgh)F(j) \arrow[dr, "\phi"] & \\
F(fg)F(h)F(j) \arrow[ur, "\phi\times 1"] \arrow[dr, "1\times\phi"'] & & F(fghj) \\
& F(fg)F(hj) \arrow[ur, "l"'] & \\
F(f)F(g)F(h)F(j) \arrow[uu, "\phi\times 1\times 1"] \arrow[dr, "1\times 1\times\phi"'] & & F(f)F(ghj) \arrow[uu, "\phi"'] \\
& F(f)F(g)F(hj) \arrow[uu, "\phi\times 1"] \arrow[ur, "1\times\phi"'] \arrow[d, equal] & \\
& F(fgh)F(j) \arrow[dr, "\phi"] & \\
F(fg)F(h)F(j) \arrow[ur, "\phi\times 1"] & & F(fghj) \\
& F(f)F(gj)F(k) \arrow[uu, "\phi\times 1"] \arrow[dr, "1\times\phi"] & \\
F(f)F(g)F(h)F(j) \arrow[uu, "\phi\times 1\times 1"] \arrow[ur, "1\times\phi\times 1"] \arrow[dr, "1\times 1\times\phi"'] & & F(f)F(ghj) \arrow[uu, "\phi"'] \\
& F(f)F(g)F(hj) \arrow[ur, "1\times\phi"'] &
\arrow[from=1-2, to=3-2, phantom, "{}^1\Downarrow\omega_{fg,h,j}"]
\arrow[from=2-1, to=5-2, phantom, "{}^3\Downarrow\cong"]
\arrow[from=3-2, to=4-3, phantom, "{}^2\Downarrow\omega_{f,g,hj}"]
\arrow[from=7-1, to=8-2, phantom, "{}^1\Downarrow\omega_{f,g,h}\times 1"]
\arrow[from=6-2, to=9-3, phantom, "{}^2\Downarrow\omega_{f,gh,j}"]
\arrow[from=8-2, to=10-2, phantom, "{}^3\Downarrow_{1\times\omega_{g,h,j}}"]
\end{tikzcd}
$$

(b) $\omega_{f,id_A,g}$, $\omega_{id_A,f,g}$ and ω_{f,g,id_A} are the identity.

Tricategories as above and unitary trihomomorphims form a category Tricat_U.

Definition 2.3.7. Let $\mathcal{T}$ be a tricategory in Tricat_U. The **nerve of a tricategory** $N_3\mathcal{T}$ is the nerve associated to the functor

$$\Delta \to \mathrm{Cat} \to Tricat_U,$$

given by considering the categories $[n]$ as tricategories with identity 2 and 3-cells. Explicitly the n-simplices of $[N_3(\mathcal{T})]_n$ are given by:

1. A collection of objects $\{X_i\}_{0\leq i\leq n}$.
2. A collection of 1-cells $\{f_{j,i} : X_i \to X_j\}_{0\leq i\leq j\leq n}$ in the bicategory $\mathcal{B}$.
3. A collection of 2-cells $\{\mu_{k,j,i} : f_{k,j} \circ f_{j,i} \Rightarrow f_{i,k}\}_{0\leq i\leq j\leq k\leq n}$.
4. A collection of 3-cells $\{\omega_{l,k,j,i}\}_{0\leq i<j<k<l\leq n}$ in $\mathcal{T}(X_i, X_l)$

$$
\begin{tikzcd}[column sep=small]
& (f_{l,k}f_{k,j})f_{j,i} \arrow[rr, Rightarrow, "a"] \arrow[dl, Rightarrow, "\mu_{l,k,j}*id_{f_{i,j}}"'] & & f_{l,k}(f_{k,j}f_{j,i}) \arrow[dr, Rightarrow, "id_{f_{l,k}}*\mu_{k,j,i}"] & \\
f_{l,k}f_{k,i} \arrow[drr, Rightarrow, "\mu_{l,j,i}"'] & & \Rrightarrow_{\omega_{l,k,j,i}} & & f_{l,j}f_{j,i} \arrow[dll, Rightarrow, "\mu_{l,k,i}"] \\
& & f_{l,i} & &
\end{tikzcd}
$$

Subject to the conditions:

1. $f_{i,i} = id_{X_i}$, for $0 \leq i \leq n$.
2. $\mu_{i,i,j} = id_{f_{i,j}}$, for $0 \leq i \leq j \leq n$.
3. $\mu_{i,j,j} = id_{f_{i,j}}$, for $0 \leq i \leq j \leq n$.
4. For $0 \leq i < j < k < l < m \leq n$, the following 3-cell diagram holds in the bicategory $\mathcal{T}(X_i, X_m)$

$$\begin{tikzcd}
\mu_{m,j,i} \circ [\mu_{m,k,j} * f_{j,i}] \circ [\mu_{m,l,k} * f_{k,j} * f_{j,i}] \arrow[dr, Rrightarrow, "\mu_{m,j,i}\circ(\omega_{m,l,k,j}*f_{j,i})"] \arrow[dd, Rrightarrow, "\omega_{m,k,j,i}\circ(\mu_{m,l,k}*f_{k,j}*f_{j,i})"'] & \\
 & \mu_{m,j,i} \circ (\mu_{m,l,k} * f_{k,i}) \circ (f_{m,l} * \mu_{l,k,j} * f_{j,i}) \arrow[dd, Rrightarrow, "\omega_{m,l,j,i}\circ(f_{m,k}*\mu_{l,k,j}*f_{j,i})"] \\
\mu_{m,k,i} \circ (f_{m,k} * \mu_{k,j,i}) \circ (\mu_{m,l,k} * f_{k,j} * f_{j,i}) \arrow[dd, Rrightarrow, "\cong"'] & \\
 & \mu_{m,l,i} \circ (f_{m,l} * \mu_{l,j,i}) \circ (f_{m,l} * \mu_{l,k,j} * f_{j,i}) \arrow[dd, Rrightarrow, "\mu_{m,l,i}\circ(f_{m,l}\mu_{l,k,j,i})"] \\
\mu_{m,k,i} \circ (\mu_{m,l,k} * f_{k,i}) \circ (f_{m,l} * f_{l,k} * \mu_{k,j,i}) \arrow[dr, Rrightarrow, "\omega_{m,l,k,i}\circ(f_{m,l}*f_{l,k}*\mu_{k,j,i})"'] & \\
 & \mu_{m,l,i} \circ (f_{m,l} * \mu_{l,k,i}) \circ (f_{m,l} * f_{l,k} * \mu_{k,j,i})
\end{tikzcd}$$

   in the above diagram any 2-cell or 1-cell should be read as the identity 3-cell corresponding to such lower cell.

The simplicial operators are induced by precomposition by the simplicial operators in Δ.

The collections are just the images of $[n]$, with n being the number of indices of the collection. The diagrams appearing in the definition are just the ones from the definition of unitary trihomomorphism, see Definition 2.3.6.

Remark 2.3.8. As in the case of the nerve of bicategories there is a geometric description for the low dimensional simplices of $N_3\mathcal{T}$. The description is the next *level* of the one presented for bicategories in Remark 2.2.12. The complete description is extensive and we omitted it here. However, we give the full description in the Appendix A. We will use such descriptions in the proof of Theorem 0.0.3 and Theorem 0.0.4.

Theorem 2.3.9. Let $\mathcal{T}$ be $(3,1)$-category then $N(\mathcal{T})$ is a quasicategory.

Proof. See [Car15, Proposition 2]. □

Theorem 2.3.9, together with Proposition 2.3.4, imply Theorem 0.0.2 stated in the introduction: the nerve of the (3,1)-category $\mathrm{Gray}_{(3,1)}$ of 2-categories, 2-functors, adjoint equivalences and invertible modification, denoted $\mathbb{G}\mathrm{ray} := N_3(\mathrm{Gray}_{(3,1)})$, is a quasicategory.

2.4 Weak n-Categories and Homotopy n-Categories

We saw that the nerve of categories, $(2,1)$-bicategories, and $(3,1)$-tricategories are quasicategories. In this section we will see that these examples also satisfy stronger inner horn lifting properties. These stronger lifting properties will serve as the starting point to define general $(n,1)$-categories. To motivate the discussion we will first consider the case of categories. A closer look on the proof of Theorem 1.2.4 gives the following:

Theorem 2.4.1. A simplicial set K is the nerve of a category C, if and only if for each $0 < i < n$ and each diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

there exist a unique dotted arrow making the diagram commutative.

Proof. The proof of 1.2.4 also proves that the nerve of a category satisfies the above property. For the converse we define a category C out of the simplicial set K. The objects of C are given by the 0-simplices of K, while the morphisms are given by the 1-simplices. The unique inner horn lifting property for Λ^2_1 will define the composition, while the unique lifting for Λ^3_2 and Λ^3_1 imply the associativity of the composition. The complete details can be found in [Dus02, Part 4]. $\square$

We will now present a definition that captures the properties of the simplicial sets considered in Theorem 2.4.1, that will allow us to state similar theorems for higher categories.

Definition 2.4.2.

1. Let $\Delta_{\leq n}$ denote the full subcategory of Δ generated by the objects $[0], \dots, [n]$. For a simplicial set the restriction to $\Delta_{\leq n}$ defines a truncation functor
$$\mathrm{tr}_n : \mathrm{Ssets} \to [\Delta_n, \mathrm{Sets}] := \mathrm{sSets}_{\leq n}\,.$$
2. The n-skeleton $\mathrm{sk}_n(K)$ of a simplicial set K has k-simplices given by $[\mathrm{sk}_n(K)]_k = K_n$, for $k \leq n$; and is freely filled with degenerate simplices for $k > n$.
3. The n-coskeleton $\mathrm{cosk}_n(K)$ of a simplicial set K has has k-simplices given by $[\mathrm{cosk}_n(K)]_k = K_n$, for $k \leq n$; while for $k = n+1$ is given by the **simplicial kernel**
$$[\mathrm{cosk}_n(K)]_k = \ker(\mathrm{tr}_k(K)).$$
Here $\ker(\mathrm{tr}_k(K))$ is the set of all n-tuples $(x_1, \dots, x_n)$, with $x_i \in [\mathrm{tr}_k(K)]_n$, satisfying the simplicial identities $\partial_i(x_j) = \partial_j(x_i)$. For $k \geq n+1$, $[\mathrm{cosk}_n(K)]_k$ is just an iterated simplicial kernel.

Informally, for every arrangement of n-simplices in $\mathrm{tr}_{\leq k}(K)$ that could be arranged as the faces of a $n+1$-simplex, there exist a unique $(n+1)$-simplex in $\ker(\mathrm{tr}_{\leq k}(K))$ with those faces. From the definition of simplicial kernel we see that if $k \geq n+1$, then a k-simplex in K must be in $\mathrm{cosk}_n(K)$. Thus we have a natural map $K \to \mathrm{cosk}_n(K)$.

Definition 2.4.3. A simplicial set K is called **n-coskeletal** if one of the following equivalent conditions holds:

1. the natural map $K \to \mathrm{cosk}_n(K)$ is an isomorphism,
2. for any other simplicial set K' the natural map
$$\mathrm{sSets}(K', K) \to [\Delta^{op}_{\leq n}, \mathrm{Sets}](\mathrm{tr}_n K', \mathrm{tr}_n K),$$

is an isomorphism,

3. for $k \geq n$ and every morphism $\partial\Delta^k \to K$ there exist a unique lifting

$$\begin{tikzcd} \partial\Delta^k \arrow[r] \arrow[d] & K \\ \Delta^k \arrow[ur, dotted] & \end{tikzcd}$$

Proof. In fact one may use the definitions of sk_n, cosk_n and $\mathrm{tr}_\leq$ to describe an adjoint triple $(\mathrm{sk}_n \dashv \mathrm{tr}_n \dashv \mathrm{cosk}_n)$, see [May92, Section II.8]. Thus properties 1 and 2 are equivalent. We now prove the equivalence between 1 and 3: the lifting property is equivalent to the existence of a unique n-simplex having a boundary determined by the image of $\partial\Delta^k$. Since such an image satisfies the simplicial identities, the definition of simplicial kernel implies the uniqueness and existence of the required n-simplex. □

With the definition of coskeletality we can restate Theorem 2.4.1 and present refinements for Theorem 2.2.13 and Theorem 2.3.9.

Theorem 2.4.4. Let K be a simplicial set. Then K is isomorphic to the nerve of some category C if and only if

1. K is a 2-coskeletal simplicial set
2. K satisfies the unique inner horn lifting property for Λ_i^n for $n = 2, 3$.

Theorem 2.4.5. Let K be a simplicial set. Then K is isomorphic to the nerve of some bicategory $\mathcal{B}$ in which every 2-cell is invertible, if and only if:

1. K is a 3-coskeletal simplicial set
2. K satisfies the unique inner horn lifting property for for $n = 3, 4$.
3. K satisfies the inner horn lifting property for $n \geq 1$.

Proof. This is proved in [Dus02, Parts 6-8], and it is the main theorem of that reference. We will give a few words on how to prove $N_2\mathcal{B}$ satisfy such properties. To begin, that $N_2\mathcal{B}$ is 3-coskeletal follows from definition of $N_2\mathcal{B}$ and the characterization of $\mathrm{cosk}(K)$ as an iterated simplicial kernel. Thus it is enough to prove the inner horn lifting properties for $n \leq 3$. This is done with arguments similar to those in the proof of Theorem 1.2.4. Basically it follows from the following informal fact: the image of an inner horn has essentially just one *combinatorial* way to be arranged, and it is determined by the image of the 1-simplices in the inner horn.

□

Theorem 2.4.6. The nerve $N_3(\mathcal{T})$ of a $(3,1)$-tricategory $\mathcal{T}$ satisfies:

1. $N_3(\mathcal{T})$ is 4-coskeletal simplicial set,
2. $N_3(\mathcal{T})$ satisfies the unique inner horn lifting property for $n = 4, 5$.
3. $N_3(\mathcal{T})$ satisfies the inner horn lifting property for $n \geq 1$.

Proof. This is [Car15, Proposition 2]. Again the ideas of the proof are similar to those in Theorems 2.4.6 and 2.4.5. The 4-coskeletality follows from the definition. The inner horn lifting conditions follows from the higher invertibility conditions defining $(3,1)$-tricategories, and these can be checked case by case for $n = 1, 2$ and 3. Property 3), Uniqueness of inner horn lifts for $n = 4, 5$, is the hardest to prove. It follows from studying the images of the inner horns for $n = 4, 5$. A careful (and ingenious) analysis of the images of the inner horns implies that there is essentially a unique way of combinatorially arranging the data of an inner horn, this determines the required unique lifting. □

Before giving a general definition of $(n,1)$-categories in the quasicategorical framework, we record two results that will be used in later chapters, when we discuss the structure of $\mathbb{E}_n$ algebras in $\mathbb{C}$at and $\mathbb{G}$ray.

Theorem 2.4.7. Let C and C' be categories. Then the nerve $N : \mathrm{Cat} \to \mathrm{sSets}$ is a fully faithful functor

Proof. Since $N(C)$ is 2-coskeletal we have

$$\mathrm{sSets}(N(C), N(C')) \cong \mathrm{sSets}_{\leq 2}(sk_2 N(C), sk_2 N(C')).$$

Using the concrete description of the first simplices, one see that the right hand side is exactly the data of a functor. □

Theorem 2.4.8. Let $\mathcal{B}$ and $\mathcal{B}'$ be 2-categories. Then the nerve $N_2 : 2\,\mathrm{Cat}_{ULax} \to \mathrm{sSets}$ is a fully faithful functor, i.e. the maps of simplicial sets $N_2\mathcal{B} \to N_2\mathcal{B}'$ can be identified with a unital pseudofunctor $\mathcal{B} \to \mathcal{B}'$.

Proof. A complete proof of this statement can be found in [BFB05, Proposition 4.3]. Here we only sketch how to recover the pseudofunctor from a map of simplicial sets. Since $N_2\mathcal{B}$ is 3-coskeletal (Theorem 2.4.5) we have

$$\mathrm{sSets}(N_2\mathcal{B}, N_2\mathcal{B}') \cong \mathrm{sSets}_{\leq 3}(sk_3 N_2\mathcal{B}, sk_3 N_2\mathcal{B}').$$

Thus we may use the explicit description of the lower dimensional simplices as in Remark 2.4.2. Considering particular instances of the simplicial assignments one may reconstruct the data of a pseudofunctor, for example the simplicial map on the 2-simplex

$$\left(\begin{tikzcd} & X \arrow[dr, "id"] \arrow[d, phantom, "\Downarrow\alpha"] & \\ Y \arrow[ur, "f"] \arrow[rr, "g"'] & {} & X \end{tikzcd} \right) \mapsto \begin{tikzcd} & F(X) \arrow[dr, "id"] \arrow[d, phantom, "\Downarrow F\alpha"] & \\ F(Y) \arrow[ur, "Ff"] \arrow[rr, "Fg"'] & {} & F(X) \end{tikzcd}$$

allows to find $F\alpha : Ff \Rightarrow Fg$ for an arbitrary 2-cell $\alpha : f \Rightarrow g$. □

Homotopy n-categories and Postnikov Systems

Following [Lur09b] and motivated by the properties of the nerves of $(1,1)$-categories, $(2,1)$-categories and $(3,1)$-tricategories, we can give a quasicategorical definition of weak $(n,1)$-categories.

Definition 2.4.9. A **weak (n,1)-category**, or just **(n,1)-category**, is a quasicategory $\mathcal{C}$ such that:

1. $\mathcal{C}$ is a $(n+1)$-coskeletal simplicial set,

2. it satisfies the unique inner horn lifting property for $m = n+1, n+2$.

We remark that, although Definition 2.4 works in general, currently there are no good explicit *algebraic* models for weak $(n,1)$-categories beyond $n = 3$. Nevertheless, Definition allows to study many abstract properties of $(n,1)$-categories even without an explicit model for them. To end this chapter we will explore some of these properties that will play an important role when discussing the stabilization hypothesis for $\mathbb{E}_n$-algebras in Chapter 4.

One important role of $(n,1)$-categories is that they form an analogue of Postnikov systems for quasicategories. For a topological space X a **Postnikov system** is a sequence of topological spaces $\tau_{\leq n}X$, called the **n-truncation** of X, each with the property that $\pi_m(\tau_{\leq n}X) = 0$ for $m > n$, and $\pi_m(\tau_{\leq n}X) = \pi_m(X)$ if $m \leq n$. These spaces come equipped with inclusions $\tau_{\leq n}X \to \tau_{\leq n-1}X$ which can be arranged into a tower

$$\cdots \to \tau_{\leq n}X \to \tau_{\leq n-1}X \to \cdots \to \tau_{\leq 2}X \to \tau_{\leq 1}X.$$

Moreover, for each n there is a canonical inclusion $\theta'_n : X \to \tau_{\leq n}X$ so that their limit induces an homotopy equivalence $X \to \lim_{\infty \leftarrow n} \tau_{\leq n}X$. Informally, Postnikov systems allow to study general topological spaces as a limit of simpler topological spaces. For a in depth discussion about Postnikov towers, we refer the reader to [Hat05, §4.3]. There is a similar sequence given by the inclusions of $(n,1)$-categories $h_{n+1}\mathcal{C}$ which will give an analogue of Postnikov systems for quasicategories.

Proposition 2.4.10. There exist an $(n,1)$-category $h_n\mathcal{C}$, called the **homotopy n-category of** $\mathcal{C}$, with the following properties:

1. There is a natural map $\theta_n : \mathcal{C} \to h_n\mathcal{C}$, which is an equivalence when $\mathcal{C}$ is an $(n,1)$-category.

2. The mapping spaces of $h_n\mathcal{C}(x,y)$ are n-truncated, i.e. $\pi_m(h_n\mathcal{C}(x,y)) = 0$ for $m \geq n$.

3. For every $(n,1)$-category $\mathcal{D}$ the mapping θ_n induces an equivalence of simplicial sets $\mathrm{sSets}(\mathcal{C},\mathcal{D}) \cong \mathrm{sSets}(h_n\mathcal{C},\mathcal{D})$.

Proof. See [Lur09b, Propositions 2.3.4.12 and 2.3.4.18] □

The homotopy n-categories of a quasicategory $\mathcal{C}$ form a system of quasicategories

$$\cdots \to h_n\mathcal{C} \to h_{n-1}\mathcal{C} \to \cdots \to h_2\mathcal{C} \to h_1\mathcal{C},$$

which we call the Postnikov system of $\mathcal{C}$. We end this chapter with a relationship between the mapping spaces of the homotopy $(n,1)$-categories and the truncation of the mapping spaces. This will allow to describe an *action* of the n-homotopy groupoid of configuration spaces when we discuss $\mathbb{E}_n$-algebras with values in a $(n,1)$-category.

Proposition 2.4.11. Let $\mathcal{C}$ be a quasicategory, $h_n\mathcal{C}$ its homotopy $(n,1)$-category and let $x, y \in \mathcal{C}_0$ be objects. Then there is an homotopy equivalence

$$\tau_{\leq n}\mathcal{C}(x,y) \cong h_n\mathcal{C}(\theta_n x, \theta_n y).$$

Proof. By the universal property of the pullback, the map of simplicial sets $\mathcal{C} \to h_n\mathcal{C}$ induces a map of simplicial sets $\phi : \mathcal{C}(x,y) \to h_n\mathcal{C}(x,y)$. Moreover, since the higher homotopy groups

$\pi_m(h_c\mathcal{C}(x,y))$ vanish for $m \geq n$, by obstruction theory (see for instance [Hat05, corollary 4.73]) there exists a lift of the map ϕ with respect to the inclusion $\iota_n : \mathcal{C} \to \tau_{\leq n}\mathcal{C}(x,y)$

$$\begin{tikzcd}
\mathcal{C}(x,y) \arrow[r, "\phi"] \arrow[d] & h_n\mathcal{C}(x,y) \\
\tau_{\leq n}\mathcal{C}(x,y) \arrow[ur, dotted, "\hat{\phi}"'] &
\end{tikzcd}$$

We claim $\hat{\phi}$ induces an isomorphism for all homotopy groups, then by Whitehead theorem it determines an homotopy equivalence of topological spaces (we can make use of Whitehead's theorem since $\mathcal{C}(x,y)$ are CW spaces). We now prove the claim by considering two cases:

1. Let $m \geq n$: This follows since both homotopy groups of $\tau_{\leq n}\mathcal{C}(x,y)$ and $h_n\mathcal{C}(x,y)$ vanish. The former by definition and the latter by Proposition 2.4.10.

2. Let $m \leq n-1$: Tracing the definitions, the map $\mathcal{C} \to h_n\mathcal{C}$ is given by

$$id \times \varphi : \Delta^0 \times_{\mathcal{C}\times\mathcal{C}} \underline{\mathrm{sSets}}(\Delta^1, \mathcal{C}) \to \Delta^0 \times_{\mathcal{C}\times\mathcal{C}} \underline{\mathrm{sSets}}(\Delta^1, h_n\mathcal{C}),$$

where $\varphi : \underline{\mathrm{sSets}}(\Delta^1, \mathcal{C}) \to \underline{\mathrm{sSets}}(\Delta^1, h_n\mathcal{C})$ is the map induced by postcomposition with $\mathcal{C} \to h_n\mathcal{C}$. Thus, it is enough to show that φ induces an isomorphism on m-homotopy groups for $m \leq n-1$. Using the definition of the sSets enrichment we have

$$\pi_m(\underline{\mathrm{sSets}}(\Delta^1, h_n\mathcal{C})) = \mathrm{sSets}(\Delta^m \times \Delta^1, h_n\mathcal{C})/\sim,$$

where the equivalence is given by homotopy relative to the $\partial\Delta^m$. Since $h_n(\Delta^m \times \Delta^1) = \Delta^m \times \Delta^1$ for $m \leq n-1$, then Proposition 2.4.10 implies that φ induces an isomorphism on m-homotopy groups for $m \leq n-1$.

□

Chapter 3

From Factorization Algebras to ∞-Operads

We will begin this chapter by presenting the physical intuition for a (pre)factorization algebra. In parallel we will motivate the definition of an operad and its possible relevance to both physics and mathematics. For this purpose we will follow the exposition of [CG21]. Factorization algebras serve as a mathematical object to model the structure of observables in a field theory, both classical and quantum. Thus to give a motivation of the definition of a factorization algebra, we have to introduce the notion of *observable* and the basic principles they should satisfy. Intuitively, an observable is anything you can quantitatively measure using a physical device which is allowed to interact with a system. From an heuristic view point observables on a classical theory should be smooth functions on a manifold $C^\infty(U)$. One may also consider other regularities for the functions and even allow the manifold to be a super or graded manifold, however smooth functions in a topological spaces is enough for the intuition. On the other hand, quantum observables should be some deformation of the classical observables $C^\infty(U)$ over a parameter $\hbar$ such that in the limit $\hbar \to 0$ we recover $C^\infty(U)$. Lets try to axiomatize and motivate some of the basic principles that an observables should satisfy.

Naive basic principles of observables:

- **Principle 1:** There exists an underlying manifold M, considered as the configuration space or the spacetime of the system, and for each open $U \subset M$ there is a set of observables $\mathrm{Obs}(U)$.
- **Principle 2 (Superposition):** We should be able to add and multiply observables, in other words $\mathrm{Obs}(U)$ should be a vector space.
- **Principle 3 (Locality):** An object, quantum or classical, is influenced directly only by its immediate surroundings. This implies that, both in the quantum an classical setting, we are allowed to make coherent simultaneous measurement as long as they are made in non related spacetime patches. Thus if $U, U' \subset V$ are disjoint there should exist a map

  $$\star : \mathrm{Obs}(U) \otimes \mathrm{Obs}(U') \to \mathrm{Obs}(V),$$

  This axiom is particularly important in quantum systems, where Heisenberg uncertainty principle implies that simultaneous measures on the same spacetime patch are incoherent.

The above discussed principles fits nicely in the definition of a prefactorization algebra.

Definition 3.0.1. Let M be a manifold. A **prefactorization algebra** $\mathcal{F}$ over M in Vec_k is the data of:

1. A vector space $\mathcal{F}$ for every open $U \subset M$, together with linear maps $\iota_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for each inclusion $U \subset V$.

2. For every finite collection of pairwise disjoint open sets $U_1, \dots U_n \subset V$ a linear map called **factorization product**

$$\mu_{U_1,\dots U_n;V} : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V).$$

The factorization product is associative in the sense that for pairwise disjoint families $\{U_{ij}\}_{j=1,\cdots,n_i} \subset V_i$ and $\{V_i\}_{i=1,\cdots,n} \subset W$, it holds

$$\begin{tikzcd}
\bigotimes_{i=1}^{n} \bigotimes_{j=1}^{n_i} \mathcal{F}(U_j) \arrow[rr, "\bigotimes_{i=1}^{n}\left(\mu_{\{U_{ij}\}_j;V_i}\right)"] \arrow[dr, "\mu_{\{U_{ij}\};W}"'] & & \bigotimes_{i=1}^{n} \mathcal{F}(V_i) \arrow[dl, "\mu_{\{V_i\};W}"] \\
& \mathcal{F}(W) &
\end{tikzcd}$$

Example 3.0.2. Let $M = \mathbb{R}$. We present a prefactorization algebra in which we can see why it is necessary to consider families of disjoint open subsets to define the factorization product:

1. For each open set $U \subset \mathbb{R}$ the factorization algebra assigns the vector space $C_c^\infty(U, T\mathbb{R})$ of compactly supported vector fields on U, i.e vector fields of the of the form $X = f\partial$ for a compactly supported function $f \in C_c(U)$. For an inclusion $U \subset V$ the compact support condition allows to extend by 0 the smooth functions on U to smooth function on V, thus there is an inclusion $C_c^\infty(U, T\mathbb{R}) \to C_c^\infty(V, T\mathbb{R})$.

2. We claim that if $U \cap V = \emptyset$, $f_1 \in C_c^\infty(U)$ and $f_2 \in C_c^\infty(V)$, then the product of vector fields is a vector field. In general this is not true, however since f_1 and f_2 have disjoint support we have,

$$\begin{aligned}
f_1\partial(f_2\partial) &= f_1(\partial f_2)\partial + f_1 f_2(\partial^2) \\
&= f_1(\partial f_2)\partial \qquad \text{(since } f_1 \text{ and } f_2 \text{ have disjoint supports).}
\end{aligned}$$

Therefore, we define the factorization product to be the product of vector fields.

We see that without the assumption that $U \cap V = \emptyset$ we could not have described the factorization product.

Remark 3.0.3. A (strict) factorization algebra is a prefactorization algebra that satisfies a *local to global* property. For completeness we state the factorization property, however it will not play a major role in this work. A **Weiss cover** for an open set U is a collection of open sets $\mathfrak{U} = \{U_i\}_{i \in I}$ such that for any finite collection of points $x_1, \dots x_k \in U$, there is an open set $U_i \in \mathfrak{U}$ containing all such points. A **factorization algebra** is a prefactorization algebra $\mathcal{F}$ that is also a cosheaf with respect to Weiss covers, i.e., for every Weiss cover the following sequence is an equalizer sequence

$$\coprod_{i,j} \mathcal{F}(U_{ij}) \rightrightarrows \coprod_{k} \mathcal{F}(U_k) \to \mathcal{F}(U),$$

where the maps are induced by the inclusions.

From both a mathematical and physical viewpoint principle 2 is too strict, intuitively we should allow any category with both a notion of addition and tensor product to make sense of observables and factorization algebras. Therefore is it natural to drop principle 2 and allow for more general tensor categories or even, as is in the case of functorial field theories, allow *higher* tensor categories. For example in the physiscs literature it is common practice to consider observables to take values in super vector spaces, to distinguish fermions and bosons observables; or in chain complexes, to allow the introduction of antifields and ghosts, etc. [Cos11, CG21]. Thus to allow such generalization it is useful to shift the focus from the observables to the structure they satisfy:

1. An indexing set given by the open sets of a manifold M.
2. Sets $\mathrm{Op}(U_1, \dots U_n; V)$ parametrising the operations we can perform on observables depending on collections of spacetimes patches $U_1, \dots U_n$ and V. In the basic setup from above this set should be just one element, encoding the factorization product, if $U_1, \dots U_n \subset V$ are pairwise disjoint, and empty otherwise.
3. Whenever $\{U_{ij}\}_{j \in J} \subset U_i$ and $\{U_i\}_{i \in I} \subset U$ there should exist an associative *composition* between the set $\mathrm{Op}(\{U_{ij}\}_{j \in J}; U_i)$ and $\mathrm{Op}(\{U_i\}_{i \in I}; U)$.

A prefactorization algebra then would be a concrete realization of the above structure on a monoidal category. We have motivated the definition of an operad (the structure of the observables of a physical system) and algebras over operads (the prefactorization algebras). We will now formalize these concepts.

3.1 Operads and Algebras over Operads

In many branches of mathematics, one might be interested in some algebraic object internal to a chosen category, for example groups on topological spaces, algebras on chain complexes, etc.. Thus it is useful to consider an external object, independent of the category in consideration, controlling the algebraic behaviour of the desired structure; this is the concept of an operad. An operad is a collection of abstract operations encoding an algebraic structure together with their relations; for example: associativity, commutativity, Jacobi identity, etc. A concrete realization of an operad on some (monoidal) category would give the desired algebraic structure on the chosen category. In this section we formally introduce the definition of an operad and their simplicial/topological enriched versions. For the general theory of operads focused on topological applications we refer the reader to [MSS02] or [Fre17]; for an algebraic perspective of operads, with emphasis on Koszul duality, we recommend [LV12] or the original work on this topic [GK94].

Definition 3.1.1. Let K be a set. A **coloured operad** $\mathcal{O}$ with colours K, denoted also $(\mathcal{O}, K)$, is the following data:

1. For every finite collection of objects $\{c_i\}_{i \in I}$, with $c_i \in K$ and indexed by a finite set I, and for every $d \in K$, a **set of operations** $\mathcal{O}(\{c_i\}_{i \in I}, d)$. We think of elements of $f \in \mathcal{O}(\{c_i\}, d)$ as single valued operation with multiple inputs $f(\{c_i\})$.
   Sometimes for clarity we will drop the indexing set and just write $\mathcal{O}(\{c_i\}, d)$.
2. A collection of **composition morphisms**

$$\prod_{j \in J} \mathcal{O}(\{b_i\}, c_j) \times \mathcal{O}(\{c_j\}, d) \to \mathcal{O}(\{b_i\}, d),$$

$$(\{f_j\}; g) \to g \circ (\{f_i\}) := g(f_1, \dots, f_j, \dots),$$

moreover the compositions are associative, in the sense that the following diagram commutes

$$\begin{tikzcd}
\prod_{j\in J}\mathcal{O}(\{b_i\},c_j)\times\prod_{k\in K}\mathcal{O}(\{c_j\},d_k)\times\mathcal{O}(\{d_k\},d) \arrow[r] \arrow[d] & \prod_{j\in J}\mathcal{O}(\{b_i\},c_j)\times\mathcal{O}(\{c_j\},d) \arrow[d] \\
\prod_{k\in K}\mathcal{O}(\{b_i\}_{i\in I},d_k)\times\mathcal{O}(\{d_k\}_{k\in K},d) \arrow[r] & \mathcal{O}(\{b_i\},d)
\end{tikzcd}$$

3. A collection of units $id_C \in \mathcal{O}(\{c\},c)$, such that for every $f \in \mathcal{O}(\{c_i\},c)$

$$id_c \circ f = f \quad \text{and} \quad f \circ \{id_c\} = f(id_c,\dots,id_c) = f.$$

An operad is called **symmetric** if it has an action of the symmetric groups. Namely for every finite set I, with $|I| = n$, and each $\sigma \in S_n$ a there is a map

$$\sigma^* : \mathcal{O}(\{c_i\}_{i\in I},c) \to \mathcal{O}(\{c_{\sigma_i}\}_{i\in I},c),$$

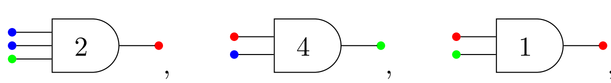
compatible with the operad structure.

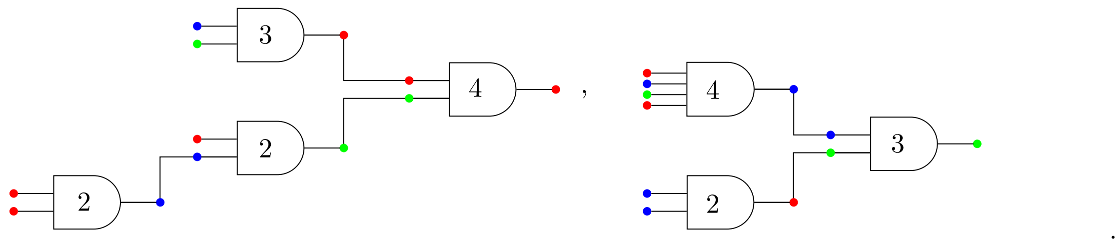
Definition 3.1.2. Let $(\mathcal{O},K)$ and $(\mathcal{O}',K')$ be colored operads. A **morphism of operads** is a map $\phi : K \to K'$ together with collection of maps

$$\phi_{\{c_i\};c} : \mathcal{O}(\{c_i\},c) \to \mathcal{O}(\{\phi(c_i)\},\varphi(c)),$$

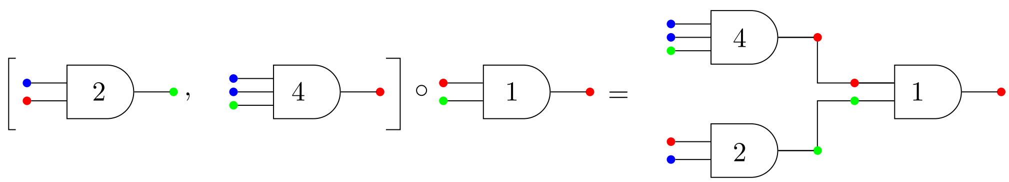
compatible with the operad structure. If the operads are symmetric we require that the maps intertwine with the action of the symmetric groups. Coloured operads together with their morphisms form a category denoted Op.

Example 3.1.3. In order to get some intuition lets consider a toy example of a (non symmetric) colored operad $\mathcal{O}$ with colors $K = \{\bullet,\bullet,\bullet\}$ and let $S = \{1,2,3,4\}$ be a labeling set. An operation in $\mathcal{O}$ will be represented by colored and labeled planar rooted tree, here we consider horizontal trees with the root on the right. Colored means we choose a color in K for each vertex in the tree, and labeled means we choose a label $S_v \in S$ for tree. We will call the initial leaves (the points in the left) of the tree as **inputs** while we call the root the **output**. Examples of elements in the operation spaces $\mathcal{O}(\bullet,\bullet,\bullet|\bullet)$, $\mathcal{O}(\bullet,\bullet|\bullet)$ and $\mathcal{O}(\bullet,\bullet|\bullet)$ are the following:





In the pictorial description the identities correspond to unbranched trees $\bullet \to \bullet$, and the composition of the above trees can be pictured as a concatenation of the trees where the input colors (on the right) match the output colors (on the left). For example



The above picture can be generalized to any sets S and K, and describes the **free coloured operad** on a (constant) set S with colors K. In general one may have different label sets for each combination of input and output colors. Free operads are generated by trees of height 1, like the top trees in the above examples, meaning any tree can be constructed by composing the height 1 trees. In general coloured operads may not be free, however given a set of relation on trees one may construct a new operad satisfying such relations. We will not formalize such description, however the reader should not find it hard to come up with a definition of quotient operads to make the idea precise (or they might just check [KV94b, Section 2.1.3])

Definition 3.1.4. An **operad** is a particular instance of a coloured operad with one colour, i.e. $K = \{*\}$. For each $n \in \mathbb{N}$ denote $\mathcal{O}(n) := \mathcal{O}(\{*\}_{1\leq i\leq n}, *)$, the compositions maps are of the form

$$\prod_{1\leq i\leq m} \mathcal{O}(n_i) \times \mathcal{O}(m) \to \mathcal{O}(n_1 + \cdots + n_m),$$

and there is just one unit element $id := id_*$. Elements of $\mathcal{O}(n)$ are called **n-ary operations**.

Example 3.1.5. We can now formalize the discussion at the beginning of the chapter and concretely describe the operad describing factorization algebras. Let M be a n-manifold. The **prefactorization operad** is the colored operad Disj_M with color set

$$\mathrm{Disk}(M) = \{U \subset M \mid U \text{ is open and } U \text{ is equal to a coproduct of disks}\},$$

and operation sets given by

$$\mathrm{Disj}_M(U_1, \dots, U_n; V) = \begin{cases} * & \text{if } U_1, \dots U_n \subset V \text{ and are pairwise disjoint} \\ \emptyset & \text{otherwise} \end{cases}$$

The units and composition are clear. The name prefactorization operad is not standard, however due to its relation with prefactorization algebras it is an appropriate name.

Example 3.1.6. We now present an classic example in the theory of operads. This example will be fundamental in the description of $\mathbb{E}_n$-algebras in Chapter 4, and in the proof of the main theorems of this thesis. The **associative operad** Ass is the operad with sets of operations

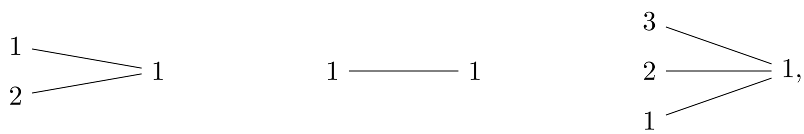
$$\mathrm{Ass}(n) := \{\leq_\sigma \ | \ \leq_\sigma \text{ is a linear ordering on } [n]\} \cong S_n, \quad \text{for } n \geq 1,$$

where the identification on the right is given by assigning to $\sigma \in S_n$ the linear ordering $i \leq_\sigma j$ if $\sigma(i) \leq \sigma(j)$ in the usual ordering of $[n]$. For $\eta \in \mathrm{Ass}(m)$ and $\tau := \{\sigma_{n_i}\} \in \prod_{1 \leq i \leq m} \mathrm{Ass}(n_i)$ the composition is given by

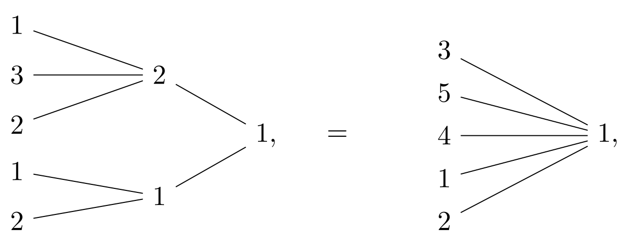
$$\prod_{1 \leq i \leq m} \mathrm{Ass}(n_i) \times \mathrm{Ass}(m) \to \mathrm{Ass}(n_1 + \cdots + n_m),$$

$$\eta \circ \tau(a_1, \ldots, a_{\sum n_i}) = \eta(\sigma_1(a_1, \ldots, a_{n_1}), \sigma_2(a_{n_1+1}, \cdots a_{n_1+n_2}), \ldots, \sigma_m(a_{N-n_m}, \ldots a_N)),$$

where $N = \sum n_i$. Element in the operations sets of Ass can be pictured by planar rooted tress labeled by sequences $(l_1, \ldots l_n)$ with $l_i \in \{1, \ldots, n\}$ and $l_i \leq l_j$ if $i \neq j$, where n is the number of inputs. One should read such tree as the unique permutation mapping $(1, \ldots, n) \mapsto (l_1, \ldots l_n)$. For example some elements in Ass(2), Ass(1) and Ass(3), respectively, are

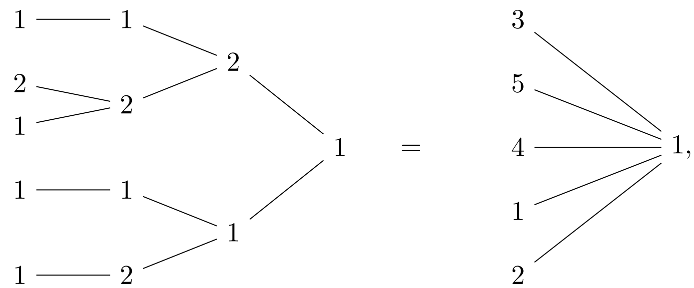


where these trees represent the permutations $(12) \in S_2$, $(1) \in S_1$, and $(321) \in S_3$ respectively. The composition gives a sequence on the total sum of the inputs, the sequence is determined by concatenating the sequences in the inputs keeping the ordering given by the outputs, as an example



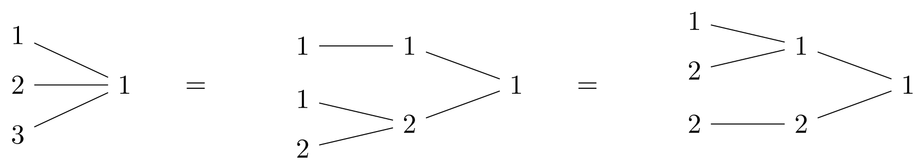
It is not hard to see that Ass is generated as an operad by $\mathrm{Ass}(2) = \{(12), (21)\}$, i.e. by the planar trees

To be precise this means any operation can be obtained as a finite composition $\mu_1 \circ \cdots \circ \mu_k$ for $\mu_i \in \mathrm{Ass}(2)$. To keep track of the order of the operad compositions, we will associate a bracketing to any decomposition of a planar tree into elements of Ass(2). For example consider the following decomposition of a tree in Ass(5)



$((\bullet(\bullet\bullet))(\bullet\bullet))$ $(\bullet\bullet\bullet\bullet\bullet)$

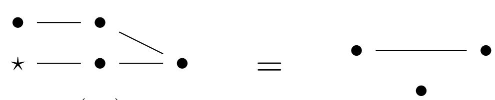
where we have described the bracketing below the tree. Let us note that this decomposition is not unique, for example the 3-ary operation given by (123) can be represented by either of the following compositions



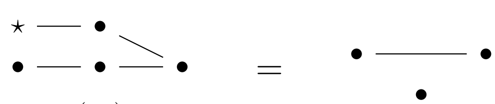
$(\bullet\bullet\bullet)$ $(\bullet(\bullet\bullet))$ $((\bullet\bullet)\bullet)$

In fact the relation above, and its permutations, span the relations of the operad Ass, see [Fre17, Proposition 1.2.7]. If we think of the decomposition in generators as bracketings, then the relation states that it does not matter how we bracket the elements, as long as we keep the same ordering. That is, this relation captures the notion of associativity.

Remark 3.1.7. To describe an operad which describes unital associative algebras we have to introduce 0-ary operations, which should correspond to $S^0 = *$. We will add a unique 0-ary operation that will be denoted by a $\star$. This 0-ary operation has a unique 1-ary relation given by the *units* relations



$(\bullet\star)$



$(\star\bullet)$

In the future when we refer to Ass we will mean this unital version.

Definition 3.1.8.

1. A **topological coloured operad** is an operad such that the sets of operations are topological spaces, which we call **operation spaces**, and the composition maps are continuous maps.

2. A **simplicial coloured operad** is an operad such that the sets of operations are simplicial spaces and the composition maps are morphisms of simplicial sets. In the case all the sets of operations are Kan simplicial sets we call the operad a **Kan coloured operad** and call the Kan complex of operations also **operation spaces**.

Definition 3.1.9. We denote by $\mathbb{D}^n$ the unit disk centered at the origin in $\mathbb{R}^n$. An embedding $f : \mathbb{D}^n \to \mathbb{D}^n$ is called **rectilinear** if it is determined by a translation and a dilation, i.e. is of the form

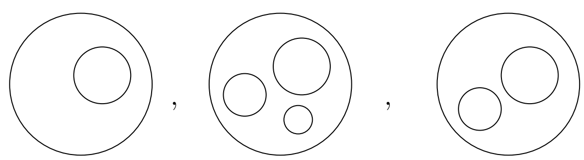
$$f(x_1, \dots, x_n) = (\lambda x_1 + t_1, \dots, \lambda x_n + t_n),$$

for some $\lambda, t_1, \dots, t_n \in \mathbb{R}$. The space of rectilinear embeddings of m disjoint disks $\mathrm{Rect}(\coprod_m \mathbb{D}^n, \mathbb{D})$ is a topological space since it can be considered as a subspace of $\mathbb{R}^{m(n+1)}$ determined by the parameters $(\lambda, t_1, \dots, t_n)$.

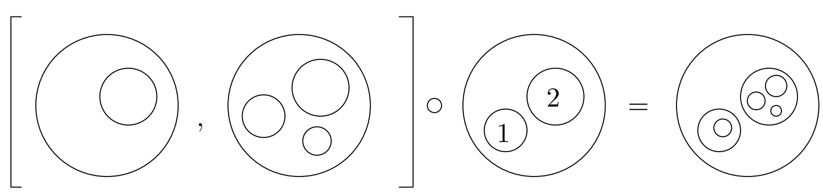
Example 3.1.10. We present now one of the main objects of study of this thesis and the operad we will mostly study in Chapter 4. The **little n-disk operad** $\mathrm{Disk}_n(m)$ is the topological operad with spaces of operations given by rectilinear embedding of disks

$$\mathrm{Disk}_n(m) := \mathrm{Rect}(\coprod_m \mathbb{D}^n, \mathbb{D}^n),$$

with units determined by the identity embeddings, and composition given by composition of embeddings. As an example we can picture operations on Disk_2 as circles embedded into a big circle



The composition can then be pictured as



Algebras over Operads

To obtain concrete algebraic structures from the *abstract* set of operations we need to *realize* this operations as morphism in some fixed category. From the definition of an (symmetric) operad one can see that to make sense of the operations we require a category with both a monoidal structure and an action of the symmetric groups. Therefore define an algebra over an operad one requires a symmetric monoidal category.

Definition 3.1.11. Let C be a symmetric monoidal category and let K be a set of colors. Then for a collection of objects $\mathbf{V} := \{V(c)\}_{c \in K}$ the **endomorphisms operad** $\mathrm{End}_{\mathbf{V}}$ is the operad with operation sets

$$\mathrm{End}_{\mathbf{V}}(\{c_i\}_{i=1,\dots,n}, c) := C(V(c_1) \otimes \cdots \otimes V(c_n), V(c)).$$

The units and compositions on the operad are induced by those in C.

Definition 3.1.12. Let $(\mathcal{O}, K)$ be a coloured operad and let C be a symmetric monoidal category. An **algebra** A **over the operad** $\mathcal{O}$ **in the category** C is the data of:

1. A collection of objects $\mathbf{V}_A := \{V(c)\}_{c\in K}$,
2. A morphism of operads $\phi_A : \mathcal{O} \to \mathrm{End}_{\mathbf{V}_A}$.

Given two algebras $(A, \mathbf{V}_A, \phi_A)$ and $(A', \mathbf{V}_{A'}, \phi_{A'})$, a **morphims of algebras over** $\mathcal{O}$ **in** C is a map of operads $\varphi : \mathrm{End}_{\mathbf{V}_A} \to \mathrm{End}_{\mathbf{V}_{A'}}$ making the following diagram commutative

$$\begin{tikzcd}
\mathrm{End}_{\mathbf{V}_A} \arrow[rr, "\varphi"] & & \mathrm{End}_{\mathbf{V}_{A'}} \\
& \mathcal{O} \arrow[ul, "\phi_A"] \arrow[ur, "\phi_{A'}"'] &
\end{tikzcd}$$

Algebras over an operad $\mathcal{O}$ in C with morphisms as above form a category which we will denote $\mathrm{Alg}_{\mathcal{O}}(C)$.

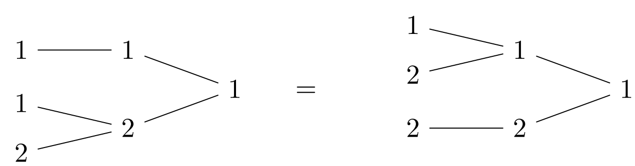
Example 3.1.13. From the discussion at the beginning of this chapter, and with the explicit definition of the prefactorization operad (Definition 3.1.5), is easy to see that a prefactorization algebra in vector spaces is the same as an algebra over the prefactorization operad in $(\mathrm{Vec}_{\mathbb{C}}, \otimes)$.

Proposition 3.1.14. An algebra over the operad Ass on $(\mathrm{Vec}_{\mathbb{C}}, \otimes)$ is an associative algebra.

Proof. The image of the unique color $*$ determines a vector space V. Since the operad Ass is generated by $(12) \in \mathrm{Ass}(2)$, then a morphism of operads $\mathrm{Ass} \to \mathrm{End}_V$ is determined by the image of (12). This image determines a product

$$\mu_2(12) =: \mu : V \otimes V \to V.$$

Moreover, the relation in Ass



$(\bullet(\bullet\bullet))$ $((\bullet\bullet)\bullet)$

implies that $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$. This is exactly the associativity condition. $\square$

3.2 A Model for ∞-Operads based in Quasicategories

We will be interested in algebraic structures up to homotopy, that is, where the constraint conditions do not hold on the nose but up to coherent homotopies. As in the strict picture, it is useful to consider a *meta* structure that upon being realized in a concrete $(\infty, 1)$-category gives the desired homotopy coherent algebraic structures. This is given by a theory of operads in the realm of $(\infty, 1)$-categories, known as ∞-operads. Let us remark that, pretty much like in the strict case, the structures of homotopy coherent algebras will (highly) depend both on the ∞-operad and the $(\infty,1)$-category we are considering the algebras on. Before giving the definition of the model of ∞-operad we will use, it is helpful to illustrate how to reinterpret the operad data into a category. This will allow us to use the framework of quasicategories developed in the first chapter to describe ∞-operads.

Definition 3.2.1.

1. The category of **pointed finite sets** Fin_* is the category with elements finite pointed sets and morphism are pointed maps. We will denote the point by $*$. The finite pointed set with n nonpointed elements is denoted by $\langle n\rangle = \{*, 1, \cdots, n\}$, where $*$ is the pointed element. When considering arbitrary finite indexing sets I we will use the notation $\langle I\rangle := I \cup \{*\}$, that is $\langle I\rangle \cong \langle |I|\rangle$.

2. Similar to the surjective-injective factorization of morphism in Sets, every morphism in Fin_* can be uniquely factorized as $g \circ f$, where f is called active and g inert. Precisely, morphisms $f, g : \langle n\rangle \to \langle m\rangle$ in Fin_* are called **active** if $f^{-1}(*) = *$, and **inert** if $g^{-1}(i)$ has exactly one element for every $i \neq *$.

   We present the fundamental examples of active and inert morphisms which will be used in later definitions. Let $\zeta_n : \langle n\rangle \to \langle 1\rangle$ be the active morphism mapping every nonpointed element to 1. On the other hand let $\rho_{(n,i)} : \langle n\rangle \to \langle 1\rangle$ be the inert morphism mapping every element except i to the point. Visualisation of these morphism can be seen in Figure 3.1, and an example of the factorization of a morphism into inert and active morphisms can be seen in Figure 3.2.

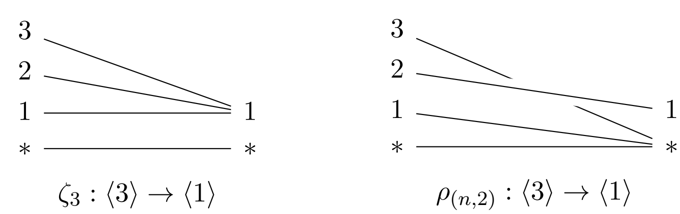


Figure 3.1: A picture for the active morphisms ζ_n and the inert morphisms $\rho_{(n,i)}$.

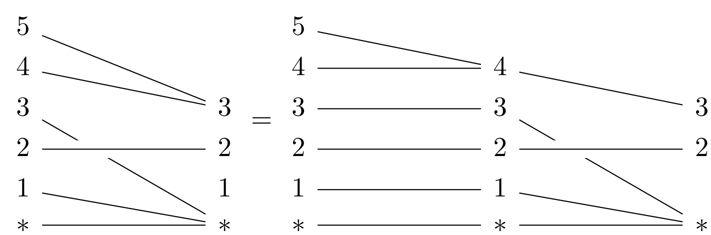


Figure 3.2: A picture for a factorisation of a morphism in Fin_*.

Definition 3.2.2. Let $(\mathcal{O}, K)$ be a coloured operad. The **category of operators** $\mathcal{O}^{\otimes}$ of $\mathcal{O}$ is the category with:

1. Objects, given by finite sequences of colours $\{c_i\}_{i\in I}$ with $c_i \in C$;

2. Morphisms sets are given by coproducts of products of operations sets, namely

$$\mathcal{O}^{\otimes}(\{c_i\}_{i\in I}, \{d_j\}_{j\in J}) = \coprod_{\phi: I\to J} \prod_{j\in J} \mathcal{O}(\{c_i\}_{i\in\phi^{-1}(j)}, d_j). \tag{3.1}$$

   Explicitly, a morphism $f : \{c_i\}_{i\in I} \to \{d_j\}_{j\in J}$ is given by a pair $(\phi, \{f_i\})$ of a map $\phi : \langle I\rangle \to \langle J\rangle$ in Fin_*, and a collection of operations

$$\{f_i\}_{i\in I} \in \prod_{i\in I} \mathcal{O}(\{c_j\}_{j\in\phi^{-1}(i)}, d_i).$$

   In words, the collection of operations have inputs labeled by the preimages of ϕ and outputs labeled by the image of ϕ. The composition and identities in $\mathcal{O}$ are induced by the operad compositions and units. Notice that in the case that the operad is a topological/simplicial operad the category of operators becomes a topological/simplicial category.

3. The category $\mathcal{O}^{\otimes}$ is equipped with a canonical forgetful functor $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$, given by $(\{c_i\}_{i\in I}) \mapsto I$ and $(\phi, \{f_i\}) \mapsto \phi$.

The importance of the category of operators is that given $\mathcal{O}^{\otimes}$ together with the functor $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$, then it is possible to recover the coloured operad $\mathcal{O}$. Intuitively, the idea is that the functor p has the information of both the color set and of the decomposition of the hom sets in Equation (3.1), and from such data one should be able to recover the colored operad structure. More precisely, following [GH15] we have the following definition and result:

Definition 3.2.3. Let $\mathrm{Cat}^{\mathrm{mult}}_{/\mathrm{Fin}_*}$ be the subcategory of $\mathrm{Cat}_{/\mathrm{Fin}_*}$ defined as follows: the objects are given by a pair of a category C and a functor $C \to \mathrm{Fin}_*$ such that:

1. There exist a cocartesian lift for all inert morphism. In particular, each inert morphism $\phi : \langle n\rangle \to \langle m\rangle$ defines a functor between fibers $\phi^! : \mathcal{O}^{\otimes}_{\langle n\rangle} \to \mathcal{O}^{\otimes}_{\langle m\rangle}$.

2. For each $n \geq 0$, the functor
$$\prod \rho^!_{(n,i)} : \mathcal{O}_{\langle n\rangle} \xrightarrow{\sim} \mathcal{O}^{\times n}_{\langle 1\rangle},$$
   induced from ${\rho_{(n,i)}}^! : \mathcal{O}^{\otimes}_{\langle n\rangle} \to \mathcal{O}^{\otimes}_{\langle 1\rangle}$ is an equivalence of categories.

3. The functor $\tilde{\rho}^!_{(n,i)} : \mathcal{O}^{\otimes}_{\phi}(C, C') \to \mathcal{O}^{\otimes}_{\rho_{(n,i)}\circ\phi}(C, C'_i)$ given by postcomposition with $\rho_{(n,i)} : C' \to C'_i$, induces an isomorphism

$$\prod_{i=1,\dots,n} \tilde{\rho}^!_{(n,i)} : \mathcal{O}^{\otimes}_{\phi}(C, C') \to \prod_{i=1,\dots,n} \mathcal{O}^{\otimes}_{\rho_{(n,i)}\circ\phi}(C, C'_i),$$

   where $\mathcal{O}^{\otimes}_{\phi}(C, C')$ are those components of $\mathcal{O}^{\otimes}(C, C')$ above ϕ.

The morphism in $\mathrm{Cat}^{\mathrm{mult}}_{/\mathrm{Fin}_*}$ are functors in $\mathrm{Cat}_{/\mathrm{Fin}_*}$ that preserve inert morphisms.

Proposition 3.2.4. The construction assigning an operad its category of operations determines an equivalence of categories $(-)^{\otimes} : \mathrm{Op} \to \mathrm{Cat}^{\mathrm{mult}}_{\mathrm{Fin}_*}$.

Proof. For complete detailed proof see [GH15, Proposition 2.2.5]. Here we only give a rough sketch on how to recover the operad structure:

- Color set K: Let $\mathcal{O}^{\otimes}_{\langle 1\rangle} := p^{-1}(\langle 1\rangle)$ be the fiber category over $\langle 1\rangle$, then the colours set can be recovered as the objects of this category.

- Operation sets: Denote the set of morphism in $\mathcal{O}^{\otimes}(\{c_i\}, c)$ over an map ϕ in Fin_* by

$$\mathcal{O}^{\otimes}_{\phi}(\{c_i\}, c) = \{f \in \mathcal{O}^{\otimes}(\{c_i\}_{i\in I}, c) \mid p(f) = \phi\}.$$

  Then operation sets can be recovered as $\mathcal{O}(\{c_i\}, c) = \mathcal{O}^{\otimes}_{\zeta_I}(\{c_i\}, c)$ (where ζ_I is as in Firgure 3.1).

- Composition: We first show how to obtain the decomposition in Equation (3.1) from the functor p. It is easy to see that any map $\phi : \langle I\rangle \to \langle J\rangle$ is the product of maps ζ_{I_j} for $I_j = \phi^{-1}(j)$, which implies

$$\mathcal{O}^{\otimes}_{\phi}(\{c_i\}, c) = \prod_{j\in j} \mathcal{O}^{\otimes}_{\zeta_{I_j}}(\{c_i\}_{i\in I_j}, c_j).$$

  Moreover, we have that $\mathrm{Fin}_*(\langle I\rangle, \langle J\rangle)$ is the coproduct over its elements. This implies the decomposition (3.1). In particular, there there are immersions

$$\prod_{j\in J} \mathcal{O}(\{b_i\}_{i\in I}, c_j) \hookrightarrow \mathcal{O}^{\otimes}(\{b_i\}_{i\in I}, \{c_j\}_{j\in J}),$$

  and the composition on the category $\mathcal{O}^{\otimes}$ induce the operad compositions

$$\begin{tikzcd}
\mathcal{O}^{\otimes}(\{b_i\}, \{c_j\}) \times \mathcal{O}^{\otimes}(\{c_i\}, d) \arrow[r] & \mathcal{O}^{\otimes}(\{b_i\}, d) \\
\prod_j \mathcal{O}(\{b_i\}, c_j) \times \mathcal{O}(\{c_j\}, d) \arrow[u, hook] \arrow[r, dotted] & \mathcal{O}(\{b_i\}, d) \arrow[u, hook]
\end{tikzcd}$$

□

Thus colored operads $(\mathcal{O}, K)$ can be modeled by categories together with a functor $\mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ satisfying the conditions in Definition 3.2.3.

Quasioperads

Definition 3.2.3 and Proposition 3.2.4 can be considered as the starting point for the definition of quasioperads, which is the model for ∞-operads based on quasicategories due to Lurie [Lur09b]. The terminology *quasioperad* is not conventional in the literature. However we will use it to make explicit that this models relies on the theory of quasicategories, as opposed to other models for ∞-operads like [CM11] or [Hau22].

Definition 3.2.5. A **quasioperad** $(\mathcal{O}^{\otimes}, p)$ is a quasicategory $\mathcal{O}^{\otimes}$ together with a map of simplicial sets $p : \mathcal{O}^{\otimes} \to N(\mathrm{Fin}_*)$ which satisfies:

1. The map p is an inner fibration with coCartesian lifts for all inert morphism (see Definition 1.3.11 and Definition 1.3.10). In particular, every inert morphism $\phi : \langle n\rangle \to \langle m\rangle$ in $N(\mathrm{Fin}_*)$ defines a functor between fibers $\phi^! : \mathcal{O}^{\otimes}_{\langle n\rangle} \to \mathcal{O}^{\otimes}_{\langle m\rangle}$.

2. For each $n \geq 0$, the functor
$$\prod \rho^!_{(n,i)} : \mathcal{O}_{\langle n\rangle} \xrightarrow{\sim} \mathcal{O}^{\times n}_{\langle 1\rangle},$$

induced from chosen lifts ${\rho_{(n,i)}}^! : \mathcal{O}^{\otimes}_{\langle n\rangle} \to \mathcal{O}^{\otimes}_{\langle 1\rangle}$ is an equivalence of quasicategories.

3. The functor $\tilde{\rho}^!_{(n,i)} : \mathcal{O}^{\otimes}_{\phi}(C, C') \to \mathcal{O}^{\otimes}_{\rho_{(n,i)}\circ\phi}(C, C'_i)$ given by postcomposition with $\rho_{(n,i)} : C' \to C'_i$, induce an equivalence of Kan complexes

$$\prod_{i=1,\dots,n} \tilde{\rho}^!_{(n,i)} : \mathcal{O}^{\otimes}{}_{\phi}(C, C') \to \prod_{i=1,\dots,n} \mathcal{O}^{\otimes}_{\rho_{(n,i)}\circ\phi}(C, C'_i),$$

where $\mathcal{O}^{\otimes}_{\phi}(C, C')$ are those components of $\mathcal{O}^{\otimes}(C, C')$ above ϕ.

Remark 3.2.6. This definition can be seen as the *requirements* to reconstruct the colored operad $\mathcal{O}$ from its category of operators $\mathcal{O}^{\otimes}$ and the forgetful functor $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$. Condition 1 is the technical replacement of the fact that p is a forgetful functor. Condition 2 ensures that, up to homotopy, objects in a quasioperad are finite sequences of colors. Condition 3 ensures that the homorphism spaces of a quasioperad have a decomposition homotopic to the decomposition in Equation (3.1).

As in the case of quasicategories, one may construct rich classes of examples for quasioperads from a nerve construction.

Definition 3.2.7. Let $\mathcal{O}$ be a topological colored operad and $\mathcal{O}^{\otimes}$ its category of operations. The **operadic nerve** $N^{\otimes}(\mathcal{O})$ of $\mathcal{O}$ is the homotopy coherent nerve of $\mathcal{O}^{\otimes}$ together with the forgetful map $Np : N(\mathcal{O}^{\otimes}) \to N(\mathrm{Fin}_*)$ induced from $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$.

Proposition 3.2.8. Let $\mathcal{O}$ be a topological operad. Then the operadic nerve $(N^{\otimes}(\mathcal{O}), Np)$ is a quasioperad.

Proof. The core of this proposition was already proved in our discussion about reconstructing $\mathcal{O}$ from $(\mathcal{O}, p)$ and in the comments in remark 3.2.6. First of all, since $\tilde{\mathcal{O}}$ is a topological category it follows, by 2.1.10, that $N(\tilde{\mathcal{O}})$ is a quasicategory. Now to check the conditions:

1. This is technical and the proof can be found in [Lur09b, Proposition 2.1.1.27.]
2. This follows from the definition of the objects in $\mathcal{O}^{\otimes}$ and the fact that $K = \mathcal{O}^{\otimes}_{\langle 1\rangle}$.
3. The decomposition of morphism spaces from the definition of $\mathcal{O}^{\otimes}$ (equation (3.1)) gives condition 3. In fact we have equality in this case, and not just homotopy equivalence.

□

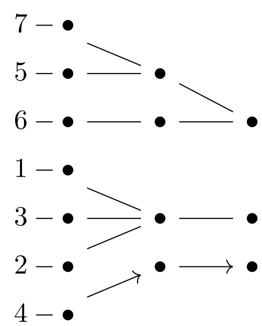
For each of the operads we have introduced in Section 3.1 we can consider its associated quasioperad via the operadic nerve construction.

Example 3.2.9. Let M be a manifold. The **prefactorization quasioperad** $\mathbb{D}\mathrm{isj}_M$ is the operadic nerve of the operad Disj_M. Since $\mathrm{Disj}^{\otimes}_M$ is a discrete Kan category, then by Lemma 2.1.12 $\mathbb{D}\mathrm{isj}_M = N(\mathrm{Disj}^{\otimes}_M)$, where on the right we mean the nerve as a category.

Example 3.2.10. The **little n-disks quasioperad** $\mathbb{E}_n$ is the operadic nerve of the topological operad Disk_n. We will come back and study this quasioperad in detail in chapter 4. For example, we will show that locally constant $\mathbb{D}\mathrm{isj}_{\mathbb{R}^n}$-algebras are equivalent to $\mathbb{E}_n$-algebras. We will also explain how $\mathbb{E}_n$-algebras give intermediate structures between homotopy associative algebras and homotopy commutative algebras.

Example 3.2.11. The **associative quasioperad** $\mathbb{A}\mathrm{ss}$ is the operadic nerve of the associative operad Ass. Again, since the operad is discrete $\mathbb{A}\mathrm{ss} = N(\mathrm{Ass}^{\otimes})$, where the nerve on the right is the nerve as a category. For the proofs of Theorems 0.0.3, 0.0.4, 0.0.5. and 0.0.6, we will need an in depth description of the quasioperad $\mathbb{A}\mathrm{ss}$.

Recall that the category of operator $\mathrm{Ass}^{\otimes}$ has objects labeled by natural numbers, and morphism spaces are given by $\coprod_{\phi:\langle m\rangle\to\langle n\rangle}\prod_{1\leq j\leq n} S_{\phi^{-1}(j)}$. Thus a morphism is given by a pair $(\phi : \langle n\rangle \to \langle m\rangle, \{\sigma_j\}_{j\in\langle m\rangle})$, where $\phi : \langle n\rangle \to \langle m\rangle$ is map in Fin_* and σ_j is a linear order in $\phi^{-1}(j)$. Thus an element in the nerve can be represented by disconnected families of labeled trees, where the roots are labeled by the codomain of the map. We call such a family a **forest**. The dimension of a simplex is represented by the length of the forest and each forest is labeled by a permutation on its input leaves. For example, a forest looks like



We will now begin to restrict the kind of forests that we need to consider in order to understand the algebras over $\mathbb{A}\mathrm{ss}$:

1. To know simplicial maps out of $\mathbb{A}\mathrm{ss}$ it is enough to consider trees, since a general forest is a coproduct of trees.

2. Since a map of simplicial sets is characterized by the images of the nondegenerate simplices, we can restrict ourselves to nondegenerate trees, i.e those for which the number of leaves always increases with the length of the tree.

3. Since the operad Ass is generated by Ass(2) and Ass(0), it follows that every tree appears as the face of a full binary tree. Thus we may restrict to full binary trees. See Figure 3.3 for an idea of how these trees look like.

4. (To be read just after Proposition 3.3.6) At last, in this thesis we will restrict ourselves to Cartesian monoidal structures. By the property of lax Cartesian morphisms we see that a permutation of the labeling of the tree is associated to a permutation of the domain of the morphism $\mathcal{C}^n \to \mathcal{C}$. Thus if we fix an order on the product, we may omit the labeling on the trees.

Therefore, to understand algebras over the quasioperad $\mathbb{A}\mathrm{ss}$ we can restrict ourselves to study the images of (unlabeled) nondegenerate binary trees. Now, the face maps in $N(\mathrm{Ass}^{\otimes})$ can be pictured as decreasing the length of the tree by merging vertices along common edges (or forgetting the initial edges). For an example of how the trees and how their faces look like see Figure 3.3. A more precise, but technical, discussion of the above statements can be found in [Lur12, Proposition 4.1.2.10].

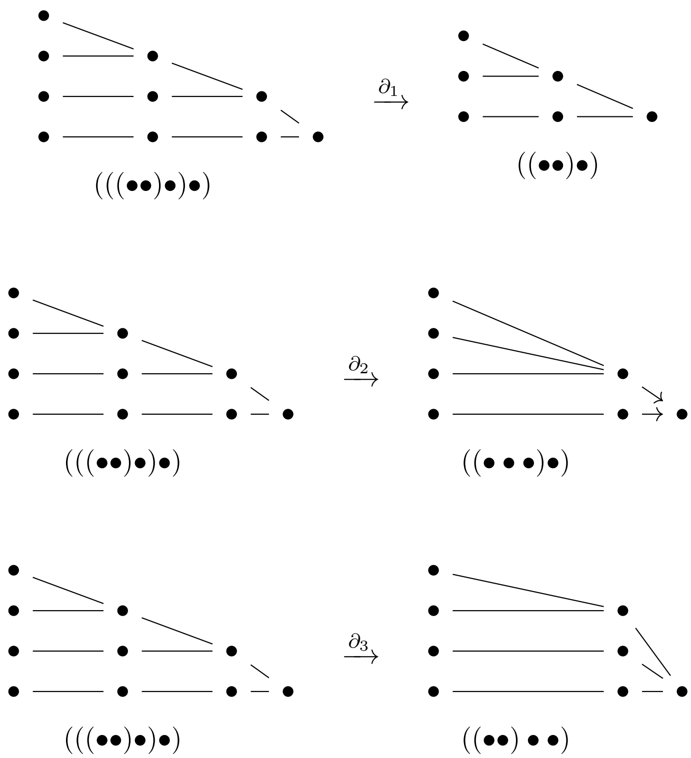


Figure 3.3: Examples of the faces of a length 3 tree.

3.3 Algebras over Quasioperads

As in the case of (strict) operads, one of the main interest is to consider algebras over them. These should model the homotopy coherent algebras we are interested in. However in order to mimic the definition of algebra from the strict setup we require a notion of symmetric monoidal quasicategory and its associated quasioperad of endomorphisms. From the operadic perspective we should be interested in the endomorphisms operad $\mathrm{End}(C)$ described by a symmetric monoidal category C, thus symmetric monoidal quasicategories should be considered as particular cases of quasioperads. We will give a definition and then give the intuition for it.

Definition 3.3.1. A **symmetric monoidal quasicategory** $(\mathcal{C}^{\otimes}, p)$ is:

1. A quasioperad $(\mathcal{C}^{\otimes}, p)$ such that,
2. the map $p : \mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$ has coCartesian lifts for all active morphism.

Equivalently, since quasioperads have cocartesian lifts for all inert morphism and every morphism has a inert/active factorization, p is a coCartesian fibration. We refer to $\mathcal{C} := \mathcal{C}^{\otimes}_{\langle 1\rangle}$ as the **underlying quasicategory** of $\mathcal{C}^{\otimes}$.

Lets try to motivate the condition 2 above. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal quasicategory, and let $\zeta^{!}_{\langle 2\rangle} : \mathcal{C}^{\otimes}_{\langle 2\rangle} \to \mathcal{C}^{\otimes}_{\langle 1\rangle}$ be a coCartesian lift of ζ_2. We thus have maps

$$\mathcal{C}^{\times n} \xleftarrow[\sim]{\prod \rho^{!}_{(n,i)}} \mathcal{C}^{\otimes}_{\langle n\rangle} \xrightarrow{\zeta^{!}_{n}} \mathcal{C},$$

and upon choosing an inverse for the equivalence $\prod \rho^!_{(n,i)}$, for which there are a contractible space of choices, we have maps $\otimes_n : \mathcal{C}^{\times n} \to \mathcal{C}$ which determine a monoidal product $\otimes_n : \mathcal{C}^{\times n} \to \mathcal{C}$. We now show how this product is homotopy associative and homotopy symmetric:

- Associativity: Since the space of cocartesian lifts is contractible we have that $\zeta^!_3$ and $\zeta^!_2 \circ (\zeta_2 \times id)$ are homotopic in $\mathrm{sSets}(\mathcal{C}_{\langle 3\rangle}, \mathcal{C})$ as both are defined by a cocartesian lift of ζ_3. Repeating the argument with the composition $\zeta_2 \circ (id \times \zeta_2)$ we have the chain of homotopies

$$(\otimes_2 \times 1) \sim \otimes_3 \sim (1 \times \otimes_2),$$

  implying homotopic associativity. Essentially the same argument gives homotopies between higher associativities given by all possible bracketings in n elements.

- Symmetry: Choose $\eta : \mathcal{C}^n \to \mathcal{C}^{\otimes}_{\langle n\rangle}$ an inverse of the equivalence given by $\prod \rho^!_{(n),i}$. Let $\sigma \in S_n$ and let $\Sigma_\sigma : \mathcal{C}^{\times n} \to \mathcal{C}^{\times n}$ be the equivalence of quasicategories given by permuting the factors of $\mathcal{C}^{\times n}$ by σ. Then $\eta \circ \Sigma_\sigma$ will also be am inverse of $\prod \rho^!_{(n),i}$, implying $\otimes_n \sim \otimes_n \circ \Sigma_\sigma$ for any $\sigma \in S_n$, this captures the homotopy symmetry.

We may now use the intuition of strict operads to give a definition of algebras over quasioperads.

Definition 3.3.2. Let $(\mathcal{O}^{\otimes}, p)$ be a quasioperad. A 1-simplex $f \in (\mathcal{O}^{\otimes})_1$ is **active** if $p(f)$ is active; and f is **inert** if $p(f)$ is inert and f is a coCartesian lift of $p(f)$.

Definition 3.3.3. Let $(\mathcal{O}^{\otimes}, p)$ and $(\mathcal{O}'^{\otimes}, p')$ be quasioperads. A **quasioperad map** is a map of simplicial sets $f : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ such that

1. It commutes with the quasioperad structure

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{\mathcal{O}}^{\otimes} \arrow[dl, "p'"] \\ & N(\mathrm{Fin}_*) & \end{tikzcd}$$

2. The map f preserves inert morphisms.

Definition 3.3.4. Let $(\mathcal{O}^{\otimes}, p)$ be a quasioperad and $\mathcal{C}^{\otimes}$ a symmetric monoidal quasicategory. An **algebra $\mathcal{A}$ over the quasioperad $\mathcal{O}^{\otimes}$ in the quasicategory** $\mathcal{C}$, or a $\mathcal{O}^{\otimes}$**-algebra in** $\mathcal{C}$, is a morphism of quasioperads $\phi_{\mathcal{A}} : \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$.

The $\mathcal{O}^{\otimes}$-algebras in $\mathcal{C}^{\otimes}$ form an quasicategory denoted $\mathrm{Alg}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes})$, namely $\mathrm{Alg}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes})$ is the full subquasicategory of $\underline{\mathrm{sSets}}_{N(\mathrm{Fin}_*)}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$ spanned by those maps that preserve inert morphisms.

In general describing a symmetric monoidal quasicategory and algebras over it is not an easy task. On one part due to the technical definitions given, and on the other part because this requires to describe morphism of simplicial sets satisfying several properties. Our approach to describe algebras over quasioperads is to consider a particular kind of symmetric monoidal quasicategory for which algebras over them are *easy* to describe.

Definition 3.3.5. Let $(\mathcal{O}, p)$ be a quasioperad and $\mathcal{C}$ a quasicategory. A **lax Cartesian functor** is a map of simplicial sets $\pi : \mathcal{O}^{\otimes} \to \mathcal{C}$ satisfying the following condition:

- Let $C \cong (C_1, \ldots, C_n)$ be a an object in $\mathcal{O}^{\otimes}_{\langle n \rangle} \cong \mathcal{C}^{\times n}$. We require that the maps $\pi_i : \pi(C) \to \pi(C_i)$ exhibits $\pi(C)$ as a product $\Pi_{1 \leq i \leq n} \pi(C_i)$ in $\mathcal{C}$.

Let $\underline{\text{sSets}}^{\text{Lax}}(\mathcal{O}^{\otimes}, \mathcal{C})$ be the full subquasicategory of $\underline{\text{sSets}}(\mathcal{O}^{\otimes}, \mathcal{C})$ spanned by lax cartesian structures. We will use the following fact:

Proposition 3.3.6. For every quasicategory $\mathcal{C}$ with finite limits, there exist a symmetric monoidal quasicategory $(\mathcal{C}^{\times}, p)$, called the **cartesian symmetric monoidal category of** $\mathcal{C}$, such that for any quasioperad $\mathcal{O}^{\otimes}$ there is an equivalence of quasicategories

$$\text{Alg}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\times}) \cong \underline{\text{sSets}}^{\text{Lax}}(\mathcal{O}^{\otimes}, \mathcal{C}).$$

Proof. See [Lur09b, §2.4] for the existence and construction of $\mathcal{C}^{\times}$. For the proof of the proposition see [Lur09b, Proposition 2.4.1.7]. For the interested reader we also discuss Cartesian symmetric monoidal quasicategories in Appendix C. □

Proposition 3.3.7. The simplicial set $\mathbb{C}\text{at}_{(2,1)}$ and $\mathbb{G}\text{ray}$ have finite products. In consequence there exist a Cartesian symmetric monoidal quasicategories $\mathbb{C}\text{at}^{\times}$ and $\mathbb{G}\text{ray}^{\times}$.

Proof. We will just prove it for $\mathbb{C}\text{at}_{(2,1)}$, since the proof for $\mathbb{G}\text{ray}_{(3,1)}$ is similar. Let I be a finite set, and let (by abuse of notation) I be the discirte simplicial set eith objects I. Then a diagram $K \to \mathbb{C}\text{at}_{(2,1)}$ is a collection $\{C_i\}_{i \in I}$ of categories indexed by the finite set I. The Cartesian product $\prod C_i$ is the category with objects $(\prod C_i)_0 = \prod (C_i)_0$ and mapping spaces $\prod C_i(\{x_i\}, \{y_i\}) = \prod C_i(x, y)$. The composition and units are defined pointwise. By definition a functor $D \to \prod C_i$ is given by a product of functors $F_i : D \to C_i$, and a natural transformation α between functors $F, G : D \to \prod C_i$ is given by a product of natural transformations $\alpha_i : F_i \to G_i$. Thus there is an equivalence of categories

$$\text{Cat}(D, \prod C_i) \to \prod \text{Cat}(D, C_1) \times \text{Cat}(D, \prod C_i).$$

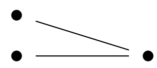
By Proposition 1.3.9, together with the fact that the nerve of categories is fully faithfull (Theorem 2.4.7), we obtain the desired result. For $\mathbb{G}\text{ray}_{(3,1)}$ instead of considering the product of hom sets one should consider the product of hom categories. □

3.4 Monoidal Categories as $\mathbb{A}$ss-algebras

To finish this chapter we will present an algebra over the quasioperads $\mathbb{A}\text{ss}$ and show how quasioperads can capture the idea of homotopic coherent algebras. Together with proposition 4.1.1 in Chapter 4, the results of this section gives a detailed proof of Theorem 0.0.3: A $\mathbb{E}_1$-algebra in $\mathbb{C}at^{\times}_{(2,1)}$ is a monoidal category. This result was already stated by Lurie in [Lur09a, Example 5.1.2.4], nevertheless here we present a detailed description that enlightens the role of quasioperads in the description of homotopic coherent algebraic structures. Moreover, our method can be naturally extended to higher $(n, 1)$-categories provided that there is a (good) nerve construction for them. We will look at the example of the quasicategory $\mathbb{G}\text{ray}$ in Chapter 4, and arrive to what is known in the literature as monoidal 2-categories.

Proof of Theorem 0.0.3. In Chapter 4 we will prove $\mathbb{A}\text{ss} \cong \mathbb{E}_1$ as quasioperads, thus describing $\mathbb{A}\text{ss}$-algebras and $\mathbb{E}_1$-algebras is equivalent. Now, by definition an $\mathbb{A}\text{ss}$-algebras on $\mathbb{C}\text{at}^{\times}$ is determined by a lax Cartesian map $\phi : \mathbb{A}\text{ss} \to \mathbb{C}\text{at}$. Denote by $C := \phi(\bullet)$ the category determined by the image of the only color in Ass. Then, ϕ is lax Cartesian if and only if $\phi(\langle n \rangle) \cong C^{\times n}$. To describe ϕ we will look at the images of the simplices in $\mathbb{A}\mathrm{ss}$, by using the explicit description given in Remark 2.2.12 and Example 3.2.11.

- 1-simplices: We consider trees of length 1. The images for these morphisms determines a monoidal product $\otimes : C \times C \to C$ and a unit $I : C^0 \to C$, where C^0 is the category with one object and one morphism.



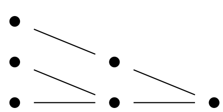
$(\bullet\bullet)$

$$\mapsto \quad C^2 \xrightarrow{\otimes} C$$

$\star$

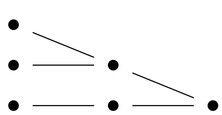
$$\mapsto \quad C^0 \xrightarrow{I} C$$

- 2-simplices: We now consider trees of length 2, and the images are given by filled triangles. We briefly explain how to get the image triangle. The vertices are labeled by the number of points, thus n points means an vertex is labeled by C^n. The edges are given by the faces in the nerve $N(\mathrm{Ass})$, thus the upper edges in the triangle are given by forgetting the first and second morphism in Ass, respectively. Meanwhile the lower edge is given by the composition of the morphism in Ass. In the below pictures these is given by the unique map $\langle 3\rangle \to \langle 1\rangle$ which we denote by F_3.



$((\bullet\bullet)\bullet)$

$$\mapsto \quad \begin{tikzcd} & C^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\alpha_1"] & \\ C^3 \arrow[ur, "\otimes\times id"] \arrow[rr, "F_3"'] & {} & C \end{tikzcd}$$



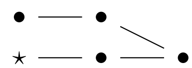
$(\bullet(\bullet\bullet))$

$$\mapsto \quad \begin{tikzcd} & C^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\alpha_2"] & \\ C^3 \arrow[ur, "id\times\otimes"] \arrow[rr, "F_3"'] & {} & C \end{tikzcd}$$

Using the assumption that α_1, α_2 are invertible natural transformations, we can invert one of these morphism to get an **associator** $\alpha := \alpha_2^{-1}\alpha_1 : \otimes \circ (\otimes \times id) \Rightarrow \otimes \circ (id \times \otimes)$

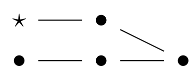
$$\begin{tikzcd} & C^2 \arrow[dr, "\otimes"] & \\ C^3 \arrow[ur, "\otimes\times id"] \arrow[dr, "id\times\otimes"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & C \\ & C^2 \arrow[ur, "\otimes"'] & \end{tikzcd}$$

Likewise we can find the **unitors** $r : \otimes \circ (id \times I) \Rightarrow id$ and $l : \otimes \circ (I \times id) \Rightarrow id$ as the images of the following trees:



$(\bullet\star)$

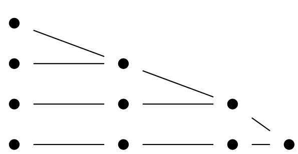
$$\mapsto \quad \begin{tikzcd} & C^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow r"] & \\ C \times C^0 \arrow[ur, "id\times I"] \arrow[rr, "id"'] & & C \end{tikzcd}$$



$(\star\bullet)$

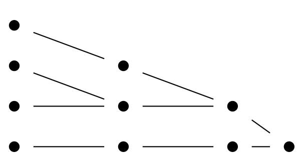
$$\mapsto \quad \begin{tikzcd} & C^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow l"] & \\ C^0 \times C \arrow[ur, "I\times id"] \arrow[rr, "id"'] & & C \end{tikzcd}$$

- 3-simplices: We now consider trees of length 3 and the images are given by filled tetahedra, which we draw as an equality of pasting diagrams.



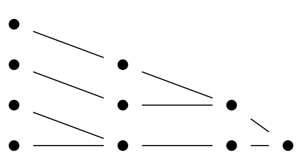
$(((\bullet\bullet)\bullet)\bullet)$

$$\mapsto \quad \begin{tikzcd} & C^3 \arrow[dr, "\otimes\times id"] \arrow[dd, "F_3"] & \\ C^4 \arrow[ur, "\otimes\times id\times id"] \arrow[dr, "F_4"'] \arrow[r, phantom, "\Leftarrow\eta_2"] & & C^2 \arrow[dl, "\otimes"] \arrow[l, phantom, "\Leftarrow\alpha_1"'] \\ & C & \end{tikzcd} = \begin{tikzcd} & C^3 \arrow[dr, "\otimes\times id"] \arrow[d, phantom, "\Downarrow \alpha_1\times id"] & \\ C^4 \arrow[ur, "\otimes\times id\times id"] \arrow[dr, "F_4"'] \arrow[rr, "F_3\times id"'] & \arrow[d, phantom, "\Downarrow \eta_2"] & C^2 \arrow[dl, "\otimes"] \\ & C & \end{tikzcd}$$



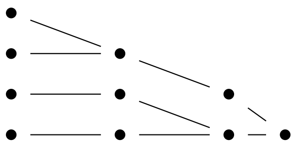
$((\bullet(\bullet\bullet))\bullet)$

$$\mapsto \quad \begin{tikzcd} & C^3 \arrow[dr, "\otimes\times id"] \arrow[dd, "F_3"] & \\ C^4 \arrow[ur, "id\times\otimes\times id"] \arrow[dr, "F_4"'] \arrow[r, phantom, "\Leftarrow\eta_3"] & & C^2 \arrow[dl, "\otimes"] \arrow[l, phantom, "\Leftarrow\alpha_1"'] \\ & C & \end{tikzcd} = \begin{tikzcd} & C^3 \arrow[dr, "\otimes\times id"] \arrow[d, phantom, "\Downarrow \alpha_2\times id"] & \\ C^4 \arrow[ur, "id\times\otimes\times id"] \arrow[dr, "F_4"'] \arrow[rr, "F_3\times id"'] & \arrow[d, phantom, "\Downarrow \eta_2"] & C^2 \arrow[dl, "\otimes"] \\ & C & \end{tikzcd}$$



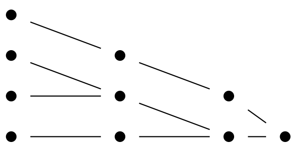
$((\bullet\bullet)(\bullet\bullet))$

$$\mapsto \quad \begin{tikzcd} & C^3 \arrow[dr, "id\times id\times\otimes"] \arrow[dd, "F_3"] & \\ C^4 \arrow[ur, "id\times id\times\otimes"] \arrow[dr, "F_4"'] \arrow[r, phantom, "\Leftarrow\eta_5"] & & C^2 \arrow[dl, "\otimes"] \arrow[l, phantom, "\Leftarrow\alpha_1"'] \\ & C & \end{tikzcd} = \begin{tikzcd} & C^3 \arrow[dr, "\otimes\times id"] \arrow[d, phantom, "\Downarrow \xi_1"] & \\ C^4 \arrow[ur, "id\times id\times\otimes"] \arrow[dr, "F_4"'] \arrow[rr, "\otimes\times\otimes"'] & \arrow[d, phantom, "\Downarrow \xi_2"] & C^2 \arrow[dl, "\otimes"] \\ & C & \end{tikzcd}$$



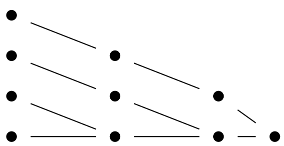
$((\bullet\bullet)(\bullet\bullet))$

$$
\begin{tikzcd}
 & C^3 \arrow[dr, "id\times\otimes"] \arrow[dd, "F_3"] & \\
C^4 \arrow[ur, "\otimes\times id\times id"] \arrow[dr, "F_4"'] \arrow[r, phantom, "\Leftarrow\eta_1"] & {} \arrow[r, phantom, "\Leftarrow\alpha_2"] & C^2 \arrow[dl, "\otimes"] \\
 & C &
\end{tikzcd}
\mapsto
\begin{tikzcd}
 & C^3 \arrow[dr, "id\times\otimes"] \arrow[d, phantom, "\Downarrow\xi_1"] & \\
C^4 \arrow[ur, "\otimes\times id\times id"] \arrow[dr, "F_4"'] \arrow[rr, "\otimes\times\otimes"'] & {} & C^2 \arrow[dl, "\otimes"] \\
 & C \arrow[u, phantom, "\Downarrow\xi_2"'] &
\end{tikzcd}
$$



$(\bullet((\bullet\bullet)\bullet))$

$$
\begin{tikzcd}
 & C^3 \arrow[dr, "id\times\otimes"] \arrow[dd, "F_3"] & \\
C^4 \arrow[ur, "id\times\otimes\times id"] \arrow[dr, "F_4"'] \arrow[r, phantom, "\Leftarrow\eta_3"] & {} \arrow[r, phantom, "\Leftarrow\alpha_2"] & C^2 \arrow[dl, "\otimes"] \\
 & C &
\end{tikzcd}
\mapsto
\begin{tikzcd}
 & C^3 \arrow[dr, "id\times\otimes"] \arrow[d, phantom, "\Downarrow id\times\alpha_1"] & \\
C^4 \arrow[ur, "id\times\otimes\times id"] \arrow[dr, "F_4"'] \arrow[rr, "F_3\times id"'] & {} & C^2 \arrow[dl, "\otimes"] \\
 & C \arrow[u, phantom, "\Downarrow\eta_4"'] &
\end{tikzcd}
$$



$(\bullet(\bullet(\bullet\bullet)))$

$$
\begin{tikzcd}
 & C^3 \arrow[dr, "id\times\otimes"] \arrow[dd, "F_3"] & \\
C^4 \arrow[ur, "id\times id\times\otimes"] \arrow[dr, "F_4"'] \arrow[r, phantom, "\Leftarrow\eta_5"] & {} \arrow[r, phantom, "\Leftarrow\alpha_2"] & C^2 \arrow[dl, "\otimes"] \\
 & C &
\end{tikzcd}
\mapsto
\begin{tikzcd}
 & C^3 \arrow[dr, "id\times\otimes"] \arrow[d, phantom, "\Downarrow id\times\alpha_2"] & \\
C^4 \arrow[ur, "id\times id\times\otimes"] \arrow[dr, "F_4"'] \arrow[rr, "F_3\times id"'] & {} & C^2 \arrow[dl, "\otimes"] \\
 & C \arrow[u, phantom, "\Downarrow\eta_4"'] &
\end{tikzcd}
$$

We can reduce the dimensionality of these pictures by considering them as 1-cells in the category $\mathbb{C}\mathrm{at}_{(2,1)}(C^4, C)$. For example, the below tetahedron is a commutative square of 2-cells

$$
\begin{tikzcd}
 & C^3 \arrow[dr, "\otimes\times id"] \arrow[dd, "F_3"] & \\
C^4 \arrow[ur, "id\times\otimes\times id"] \arrow[dr, "F_4"'] \arrow[r, phantom, "\Leftarrow\eta_3"] & {} \arrow[r, phantom, "\Leftarrow\alpha_1"] & C^2 \arrow[dl, "\otimes"] \\
 & C &
\end{tikzcd}
=
\begin{tikzcd}
 & C^3 \arrow[dr, "\otimes\times id"] \arrow[d, phantom, "\Downarrow \alpha_2\times id"] & \\
C^4 \arrow[ur, "id\times\otimes\times id"] \arrow[dr, "F_4"'] \arrow[rr, "F_3\times id"'] & {} & C^2 \arrow[dl, "\otimes"] \\
 & C \arrow[u, phantom, "\Downarrow\eta_2"'] &
\end{tikzcd}
$$

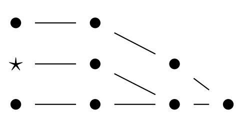
$$
\begin{tikzcd}
H_{2,1} \arrow[r, "\alpha_1*(id\times\otimes\times id)"] \arrow[d, "\otimes*(\alpha\times id)"'] & F_3\circ(id\times\otimes\times id) \arrow[d, "\eta_3"] \\
\otimes\circ(F_3\times id) \arrow[r, "\eta_2"'] & F_4
\end{tikzcd}
$$

where $H_{2,1}$ is the composition $\otimes\circ(\otimes\times id)\circ(id\times\otimes\times id)$. We will generally omit the vertices of the diagrams when doing such reductions, since they can be read from the pasting diagrams which define them. Using this reduction in dimension we can arrange the above tetrahedrons, which now become squares, into a pentagon diagram. Upon using the definition of the associator $\alpha = \alpha_2^{-1}\alpha_1$, and using the inverses of the commutative squares, we can arrange all the data to construct **Mac Lane's pentagon diagram**.

$$\begin{tikzcd}
 & & (((\bullet\bullet)\bullet)\bullet) & & \\
((\bullet\bullet)(\bullet\bullet)) \arrow[urr, "\alpha_1^{-1}\alpha_2"] & & & & ((\bullet(\bullet\bullet))\bullet) \arrow[ull, "\alpha_1^{-1}\alpha_2\times id"'] \\
 & & F_4 & & \\
 & (\bullet(\bullet(\bullet\bullet))) \arrow[uul, "\alpha_1^{-1}\alpha_2"] \arrow[rr, "id\times\alpha_1^{-1}\alpha_2"'] & & (\bullet((\bullet\bullet)\bullet)) \arrow[uur, "\alpha_1^{-1}\alpha_2"'] &
\end{tikzcd}$$

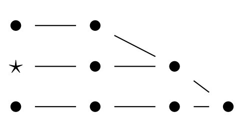
η_1 η_2 η_3 η_4 η_5

On the other hand, to get the the coherence for the right and left unitors consider



$(\bullet(\star\bullet))$

$$\mapsto \begin{tikzcd}
 & C^3 \arrow[dr, "id\times\otimes"] \arrow[dd, "F_3"] & \\
C_\times C^0{}_\times C \arrow[ur, "id\times I\times id"] \arrow[dr, "\otimes"'] \arrow[r, phantom, "\Leftarrow\zeta"] & \arrow[r, phantom, "\Leftarrow\alpha_1"] & C^2 \arrow[dl, "\otimes"] \\
 & C &
\end{tikzcd} = \begin{tikzcd}
 & C^3 \arrow[dr, "id\times\otimes"] \arrow[d, phantom, "\Downarrow id\times l"] & \\
C_\times C^0{}_\times C \arrow[ur, "id\times I\times id"] \arrow[rr, "id"'] \arrow[dr, "\otimes"'] & {} \arrow[d, phantom, "="] & C^2 \arrow[dl, "\otimes"] \\
 & C &
\end{tikzcd}$$



$((\bullet\star)\bullet)$

$$\mapsto \begin{tikzcd}
 & C^3 \arrow[dr, "\otimes\times id"] \arrow[dd, "F_3"] & \\
C_\times C^0{}_\times C \arrow[ur, "id\times I\times id"] \arrow[dr, "\otimes"'] \arrow[r, phantom, "\Leftarrow\zeta"] & \arrow[r, phantom, "\Leftarrow\alpha_2"] & C^2 \arrow[dl, "\otimes"] \\
 & C &
\end{tikzcd} = \begin{tikzcd}
 & C^3 \arrow[dr, "\otimes\times id"] \arrow[d, phantom, "\Downarrow r\times id"] & \\
C_\times C^0{}_\times C \arrow[ur, "id\times I\times id"] \arrow[rr, "id"'] \arrow[dr, "\otimes"'] & {} \arrow[d, phantom, "="] & C^2 \arrow[dl, "\otimes"] \\
 & C &
\end{tikzcd}$$

Which again can be reduced in dimension when we consider the above as 1-cells in $\mathbb{C}\mathrm{at}(C \times C^0 \times C, C)$

$$\begin{tikzcd}
\otimes\circ(\circ\times id) \arrow[r, "\alpha_1"] \arrow[dr, "id\times l"'] & F_3 \arrow[d, "\zeta"] & \otimes\circ(id\times\circ) \arrow[l, "\alpha_2"'] \arrow[dl, "id\times r"] \\
 & \otimes &
\end{tikzcd}$$

which, again inverting the commutative square with α_2, gives the coherence between unitors.

We conclude that an $\mathbb{E}_1$-algebra in $\mathbb{C}\mathrm{at}^\times$ determines a monoidal category. Moreover given a monoidal category $(C', \otimes', \alpha')$ one can build a a $\mathbb{E}_1$-algebra by setting most of the above data to be the identity; for example in the above notation one should set $\alpha_1 := \alpha'$, $\alpha_2 := id$ and just of the η_1 and η_2 to be nondegenerate. $\square$

Chapter 4

$\mathbb{E}_n$-Algebras on Low Dimensions

The purpose of this chapter is to present the main results to describe $\mathbb{E}_n$-algebras on $(m,1)$-categories for low m. In particular we prove the main theorems of this work: Theorem 0.0.3, Theorem 0.0.4, Theorem 0.0.5 and Theorem 0.0.6. We begin this chapter by showing that locally constant (pre)factorization algebras over $\mathbb{R}^n$ are equivalent to $\mathbb{E}_n$-algebras. Thus connecting the title of this thesis with its main results. To avoid introducing additional concepts, that would obscure the main idea of the argument, we will restrict the discussion to the case where the base manifold is $\mathbb{R}^n$. However we remark that the argument works for more general manifolds equipped with a tangential structure. A reader interested in locally constant factorization algebras for general manifolds, the relationship with tangential structures and variations of the $\mathbb{E}_n$-operad, may consult [AF15] section 2 or [Lur09a] sections 3.1 and 3.2. We begin with a formal definition of locally constant prefactorization algebras.

Definition 4.0.1. A **locally constant prefactorization algebra in** $\mathbb{R}^n$ on a symmetric monoidal quasicategory $\mathcal{C}^{\otimes}$ is a $\mathbb{D}\mathrm{isj}_n := \mathbb{D}\mathrm{isj}_{\mathbb{R}^n}$-algebra on $\mathcal{C}^{\otimes}$

$$F : \mathbb{D}\mathrm{isj}_n \to \mathcal{C}^{\otimes},$$

such that for every inclusion $j = (U \subset V \subset \mathbb{R}^n)$, seen as a 1-cell in $\mathbb{D}\mathrm{isj}_n$ over $id_{\langle 1\rangle}$, the map $F(j)$ is an equivalence in $\mathcal{C}$.

The idea to prove the equivalence of locally constant (pre)factorization algebras and E_n-algebras is to relate the quasioperads $\mathbb{D}\mathrm{isj}_n := \mathbb{D}\mathrm{isj}_{\mathbb{R}^n}$ and $\mathbb{E}_n$ of examples 3.2.9 and 3.2.10. For this we will start by introducing a quasioperad equivalent to $\mathbb{D}\mathrm{isj}_n$ that will allow us to make a comparison to $\mathbb{E}_n$ more easily.

Definition 4.0.2. Let $\mathbb{D}\mathrm{isj}'_n$ be the colored operad with color set

$$\mathrm{Disk}(M) = \{j : \mathbb{D}^n \to \mathbb{R}^n \mid j \text{ is a rectilinear embedding}\},$$

and operation sets $\mathrm{Disj}'_n(\{j_1, \dots, j_m\}, j)$ given by commuting diagrams of rectilinear embeddings

$$\begin{tikzcd} \coprod_m \mathbb{D}^n \arrow[rr] \arrow[dr, "\coprod_m j_i"'] & & \mathbb{D}^n \arrow[dl, "j"] \\ & \mathbb{R}^n & \end{tikzcd}, \tag{4.1}$$

such that the images of j_i are disjoint. The compositions are given by composing the horizontal embeddings in 4.1, and units are given by the identity embedding.

Since rectilinear embeddings are characterized by their image we can conclude the following: if the images are disjoint and $\mathrm{im}(j_i) \subset \mathrm{im}(j)$, then $\mathrm{Disj}'_n(\{j_1, \ldots, j_m\}, j)$ has only one element; furthermore $\mathrm{Disj}'_n(\{j_1, \ldots, j_m\}, j)$ is empty otherwise. Define a map of operads $\psi : \mathrm{im} : \mathrm{Disj}'_n \to \mathrm{Disj}_n$ by the following data:

1. On color sets, it sends an rectilinear embedding $j : \mathbb{D} \to \mathbb{R}^n$ to its image $\mathrm{im}(j) \subset \mathbb{R}^n$.
2. On operation spaces, it sends the unique commutative diagram as in 4.1 to the unique element of $\mathrm{Disj}(\mathrm{im}(j_i), \ldots, \mathrm{im}(j_n); \mathrm{im}(j))$.

Lemma 4.0.3. The map of operads $\mathrm{im} : \mathrm{Disj}'_n \to \mathrm{Disj}$ is an equivalence of operads. Therefore, there is an equivalence of quasioperads $N(\mathrm{im}) : \mathbb{D}\mathrm{isj}'_n \to \mathbb{D}\mathrm{isj}_n$.

Proof. Since rectilinear embeddings are characterized by their image we have that ψ induces an isomorphism on colors. Moreover, since on both cases operation spaces are given by one objects or the empty set (given by the same conditions), then we see that it also determines isomorphisms of operation spaces compatible with the operad structure. □

Using the explicit description of the low dimensional simplices in the homotopy coherent nerve from example 2.1.11, we can see that 1-simplices of $\mathbb{E}_n$ are given by (coproducts of) rectilinear embeddings $j : \coprod_m \mathbb{D}^n \to \mathbb{R}^n$, while 2-simplices are given by (coproducts of) pairs of embeddings $j_1, j_2 : \coprod_m \mathbb{D}^n \to \mathbb{R}^n$ and a homotopy between them

$$\begin{tikzcd} \coprod_m \mathbb{D}^n \arrow[rr] \arrow[dr, "j_1"'] & \sim & \coprod_m \mathbb{D}^n \arrow[dl, "j_2"] \\ & \mathbb{R}^n & \end{tikzcd} \tag{4.2}$$

Thus the objects in the operads $\mathbb{E}_n$ and $\mathbb{D}\mathrm{isj}'_n$ are the same, however their morphisms are different. On one side, morphism of $\mathbb{D}\mathrm{isj}'_n$ are commutative diagrams as in 4.1, while morphisms in $\mathbb{E}_n$ are homotopy commutative diagrams as in 4.2. Informally, $\mathbb{E}_n$ is a *topologised* version of $\mathbb{D}\mathrm{isj}'_n$. The important fact is that the inclusion $\mathbb{D}\mathrm{isj}'_n \to \mathbb{E}_n$ becomes an equivalence of quasicategories after *inverting* the diagrams in $\mathbb{D}\mathrm{isj}'_n$ that are homotopy equivalences.

Theorem 4.0.4 ([AF15])**.** Let $\mathcal{I}_n \subset \mathbb{D}\mathrm{isj}'_n$ be the quasicategory spanned by the same objects of $\mathbb{D}\mathrm{isj}_n$ but only those morphisms whose image in $\mathbb{E}_n$ are equivalences. Then then the pair $\mathbb{E}_n$ and $\mathbb{D}\mathrm{isj}_n \to \mathbb{E}_n$ is a localization of quasicategories (see Definition 1.3.2).

Proof. See [AF15] Proposition 2.19. □

Corollary 4.0.5 ([Lur09a])**.** For every symmetric monoidal quasicategory $\mathcal{C}^\otimes$ the map of simplicial sets $\mathbb{D}\mathrm{isj}_n \to \mathbb{E}_n$ induces a map of quasicategories $\mathrm{Alg}(\mathbb{E}_n) \to \mathrm{Alg}(\mathbb{D}\mathrm{isj}_n)$, whose image is the full subquasicategory spanned by the locally constant (pre)factorization algebras.

Proof. An algebra over the operad $\mathbb{D}\mathrm{isj}_n$ lies in the image of $\mathrm{Alg}(\mathbb{E}_n) \to \mathrm{Alg}(\mathbb{D}\mathrm{isj}_n)$ if and only if it can be factorized as

$$\begin{tikzcd} \mathbb{D}\mathrm{isj}_n \arrow[d] \arrow[dr] & \\ \mathbb{E}_n \arrow[r] & \mathcal{C}^\otimes \end{tikzcd}$$

By Theorem 4.0.4 and the definition of a localization, such a factorization occurs if and only if, 1-cells in $\mathcal{I}_n$ are mapped to equivalences in $\mathcal{C}^\otimes$. Since any two embeddings $\iota_1, \iota_2 : \coprod \mathbb{D}^n \to \mathbb{R}^n$

are homotopic ($\mathbb{R}^n$ is contractible) we have that these are exactly all prefactorization algebras such that for every $\alpha = (U \subset V)$ the map $\mathcal{F}(\alpha)$ is an equivalence. These are exactly the locally constant factorization algebras. $\square$

Thus $\mathbb{E}_k$-algebras are a very specific cases of general factorization algebras. Nevertheless, as we will see, the study and description of $\mathbb{E}_n$-algebras is not trivial and actually considering them on $(n,1)$-categories gives a new light to many objects in higher category theory and representation theory.

4.1 Tensor product of Quasioperads and Dunn's Additivity

We will now embark our study of the structure of $\mathbb{E}_n$ algebras on $(m,1)$-categories for low m.

$\mathbb{E}_1$-algebras as Associative Algebras

Proposition 4.1.1. There is an equivalence of quasioperads $\mathbb{E}_1 \cong \mathbb{A}\mathrm{ss}$. We conclude that for any symmetric monoidal quasicategory $\mathcal{C}^{\otimes}$, there is an equivalence of quasicategories

$$\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}^{\otimes}) \cong \mathrm{Alg}_{\mathbb{A}\mathrm{ss}}(\mathcal{C}^{\otimes}).$$

Proof. We will identify $\mathbb{D}_1$ with $(0,1)$. Every rectangular embedding $f : \coprod_m (0,1) \to (0,1)$ determines and is determined, up to homotopy, by a linear ordering on the set $\{0, \dots, m-1\}$. Indeed, one can define $i < j$ if $f_i(t) < f_j(t)$ for all $t \in (0,1)$, where f_m is the m-th component of the embedding f. For an example of this maps see Figure 4.1.

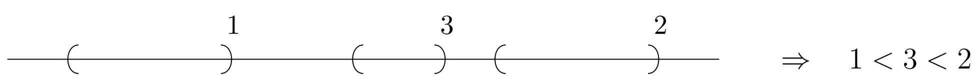


Figure 4.1: Ordering on $\{1,2,3\}$ determined by an embedding $\coprod_{1,2,3} \mathbb{D}^1 \to \mathbb{D}^1$.

The above assignment induce a continuous map $\Sigma : \mathrm{Rect}(\coprod_m (0,1), (0,1)) \to S_m$ with contractible fibers. Therefore we conclude there is a homotopy equivalence

$$\mathrm{Rect}(\coprod_m (0,1), (0,1)) \cong S_n,$$

which is compatible with the corresponding operads structure. These homotopy equivalences induces a weak equivalence of topological operads $\mathrm{Disk}_1 \xrightarrow{\sim} \mathrm{Ass}$, which upon applying the homotopy coherent nerve gives our desired equivalence $\mathbb{E}_1 \xrightarrow{\sim} \mathbb{A}\mathrm{ss}$. $\square$

Tensor product of Quasioperads

The second result to study the structure of $\mathbb{E}_n$-operads on $(n,1)$-categories is Dunn's additivity, which characterizes $\mathbb{E}_{k+1}$-algebras as $\mathbb{E}_1$-algebras on $\mathbb{E}_k$-algebras on $\mathcal{C}$. To state properly this theorem we should first discuss the tensor product of quasioperads. For this we will follow [Lur09b].

Definition 4.1.2. The **wedge product** of pointed finite sets $\wedge : \mathrm{Fin}_* \times \mathrm{Fin}_* \to \mathrm{Fin}$ as follows:

1. On objects, it is given by $\langle m \rangle \wedge \langle n \rangle = \langle mn \rangle$.

2. On morphism, it maps a tuple $f : \langle n\rangle \to \langle n'\rangle$ and $g : \langle m\rangle \to \langle m'\rangle$ to the map $f \wedge g$ determined by

$$f \wedge g(\mu_1 + \mu_2 n - n) = \begin{cases} * , & \text{if } f(\mu_1) = * \text{ or } g(\mu_2) = *, \\ f(\mu_1) + g(\mu_2)n' - n' & \text{otherwise,} \end{cases} \tag{4.3}$$

where $1 \leq \mu_1 \leq n$ and $1 \leq \mu_2 \leq m$.

Example 4.1.3. The definition of the the wedge product will be important when we discuss $\mathbb{E}_2$-algebras, thus here we will show an example of (4.4). Consider

$$\begin{tikzcd} 2 \arrow[rr] & & 2 \\ 1 \arrow[rr] & & 1 \\ * \arrow[rr] & & * \end{tikzcd} \qquad \begin{tikzcd} 2 \arrow[drr] & & \\ 1 \arrow[rr] & & 1 \\ * \arrow[rr] & & * \end{tikzcd}$$

$$f : \langle 2\rangle \to \langle 2\rangle \qquad\qquad g : \langle 2\rangle \to \langle 1\rangle$$

Then the wedge products $f \wedge g$ and $g \wedge f$ are given by

$$\begin{tikzcd} 4 \arrow[drr] & & \\ 3 \arrow[ddrr] & & 2 \\ 2 \arrow[urr] & & \\ 1 \arrow[rr] & & 1 \\ * \arrow[rr] & & * \end{tikzcd} \qquad \begin{tikzcd} 4 \arrow[drr] & & \\ 3 \arrow[rr] & & 2 \\ 2 \arrow[drr] & & \\ 1 \arrow[rr] & & 1 \\ * \arrow[rr] & & * \end{tikzcd} \tag{4.4}$$

$$f \wedge g : \langle 4\rangle \to \langle 2\rangle \qquad\qquad g \wedge f : \langle 4\rangle \to \langle 2\rangle$$

This will later be associated to the appearance of a braiding on $\mathbb{E}_2$-algebras. Loosely speaking, morphism on the left of 4.4 will be associated to a product $(X_1 \otimes X_2, X_3 \otimes X_4)$, while the morphism on the right of 4.4 will be associated to a product $(X_1 \otimes X_3, X_2 \otimes X_4)$. However, after postcomposing with the unique product over the morphism $\langle 2\rangle \to \langle 1\rangle$ both of this products should be homotopic to the unique product associated to the morphism $\langle 4\rangle \to \langle 1\rangle$. Thus we will have an homotopy $X_1 \otimes X_2 \otimes X_3 \otimes X_4 \sim X_1 \otimes X_3 \otimes X_2 \otimes X_4$. We will make this discussion rigorous after formally stating Dunn's additivity theorem (Theorem 4.1.8).

Example 4.1.4. One important case we need to consider are the wedges of the inert morphisms $\rho_{(n,i)} \wedge \rho_{(m,j)} : \langle nm\rangle \to \langle 1\rangle$. We have

$$\rho_{(n,i)} \wedge \rho_{(m,j)}(\mu_1 + \mu_2 n - n) = \begin{cases} 1, & \text{for } \mu_1 = i \text{ and } \mu_2 = j, \\ * & \text{otherwise,} \end{cases} \tag{4.5}$$

where $1 \leq \mu_1 \leq n$ and $1 \leq \mu_2 \leq m$. Intuitively, if we divide nm into m blocks of n elements, then $\rho_{(n,i)} \wedge \rho_{(m,j)}$ picks the ith-elements in the jth-block. We remark that $\rho_{(n,i)} \wedge \rho_{(m,j)}$ and $\rho_{(m,i)} \wedge \rho_{(n,j)}$ are different. For example, let $\langle nm\rangle = \{1, 2, 3, 4, 5, 6, *\}$, then we can tabulate the unique element mapping to 1 for different wedge products:

	$(i=1, j=1)$	$(i=1, j=2)$	$(i=1, j=3)$	$(i=2, j=1)$	$(i=2, j=2)$	$(i=2, j=3)$
$\rho_{(2,i)} \wedge \rho_{(3,j)}$	1	3	5	2	4	6

	$(i=1,j=1)$	$(i=1,j=2)$	$(i=2,j=1)$	$(i=2,j=2)$	$(i=3,j=1)$	$(i=3,j=2)$
$\rho_{(3,i)} \wedge \rho_{(2,j)}$	1	4	2	5	3	6

Definition 4.1.5. Let $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ be quasioperads. A **bifunctor** of quasioperads is a map of simplicial sets $\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ such that:

1. the following diagram commutes,

$$\begin{array}{ccc} \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} & \longrightarrow & \mathcal{O}''^{\otimes} \\ \downarrow & & \downarrow \\ N(\mathrm{Fin}_*) \times N(\mathrm{Fin}_*) & \xrightarrow{\wedge} & N(\mathrm{Fin}_*) \end{array}$$

2. for every pair of inert morphisms $\alpha \in \mathcal{O}^{\otimes}_1$ and $\beta \in \mathcal{O}'^{\otimes}_1$, the image $f(\alpha, \beta)$ is a inert morphism in $\mathcal{O}''^{\otimes}$.

Denote by $\mathrm{BiFunc}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}; \mathcal{O}''^{\otimes})$ the full subquasicategory of $\underline{\mathrm{sSets}}_{/N(\mathrm{Fin}_*)}(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes})$ spanned by the bifunctors.

Definition 4.1.6. Let $\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ be a bifunctor. We say $(\mathcal{O}''^{\otimes}, f)$ is an **operadic tensor product**, if for every quasioperad $\mathcal{C}^{\otimes}$ precomposition with f induces an equivalence of quasicategories

$$\mathrm{Alg}_{\mathcal{O}''^{\otimes}}(\mathcal{C}^{\otimes}) \xrightarrow{\sim} \mathrm{BiFunc}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}; \mathcal{C}^{\otimes}).$$

Remark 4.1.7. If $(\mathcal{O}^{\otimes}, p)$ and $(\mathcal{O}'^{\otimes}, p')$ are quasioperads, then by the definition of quasioperad there is a natural isomorphism

$$\prod_{1 \le i \le n,\, 1 \le j \le m} (\rho_{(n,i)} \wedge \rho_{(m,j)}) : \mathcal{O}^{\otimes}_{\langle n \rangle} \times \mathcal{O}'^{\otimes}_{\langle m \rangle} \xrightarrow{\sim} \prod_{k=1}^{nm} (\mathcal{O}^{\otimes}_{\langle 1 \rangle} \times \mathcal{O}'^{\otimes}_{\langle 1 \rangle}) \cong \prod_{k=1}^{nm} (\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes})_{\langle 1 \rangle}.$$

Thus a general element in $\mathcal{O}^{\otimes}_{\langle n \rangle} \times \mathcal{O}'^{\otimes}_{\langle m \rangle}$ is a product of nm elements in $(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes})_{\langle 1 \rangle}$. Notice that the *order* of this product depends on the order of n and m, since in general $\rho_{(n,i)} \wedge \rho_{(m,j)}$ is different from $\rho_{(m,i)} \wedge \rho_{(n,i)}$ (See Example 4.1.4).

Dunn's Additivity

Our goal is to construct a bifuntor of quasioperads $f : \mathbb{E}_k \times \mathbb{E}_{k'} \to \mathbb{E}_{k+k'}$ so that $(\mathbb{E}_{k+k'}, f)$ is a tensor product. To begin we will consider the functor of topological categories

$$\times : \mathrm{Disk}^{\otimes}_k \times \mathrm{Disk}^{\otimes}_{k'} \to \mathrm{Disk}^{\otimes}_{k+k'},$$

defined as follows:

1. On objects, which are canonically isomorphic to Fin_* since the color set is a point: the functor is given by the wedge product $\wedge : \mathrm{Fin}_* \wedge \mathrm{Fin}_* \to \mathrm{Fin}_*$.

2. Let $(\alpha, \phi) \in \mathbb{E}_k(\langle m \rangle, \langle n \rangle)$ and $(\beta, \psi) \in \mathbb{E}_k(\langle m \rangle, \langle n \rangle)$ be morphism. Explicitly, $\phi : \langle m \rangle \to \langle n \rangle$ and $\psi : \langle m' \rangle \to \langle n' \rangle$ are maps in Fin_*, and

$$\alpha \in \prod_{1 \le j \le n} \mathrm{Rect}(\coprod_{\phi^{-1}(j)} \mathbb{D}^k, \mathbb{D}^k) \quad \text{and} \quad \beta \in \prod_{1 \le j \le n} \mathrm{Rect}(\coprod_{\psi^{-1}(j)} \mathbb{D}^k, \mathbb{D}^k).$$

We define the product $(\alpha, \phi) \times (\beta, \psi) = (\alpha \times \beta, \phi \wedge \psi)$, where $\alpha \times \beta$ is given by component wise product, i.e.

$$(\alpha \times \beta)_{i,i'} = \alpha_i \times \beta_{i'} : \coprod_{\phi^{-1}(i) \times \psi^{-1}(i)} \mathbb{D}^{k+k'} \to \mathbb{D}^{k+k'}.$$

Here we used a fixed homeomorphism $\mathbb{D}^k \times \mathbb{D}^k \cong \mathbb{D}^{k+k'}$.

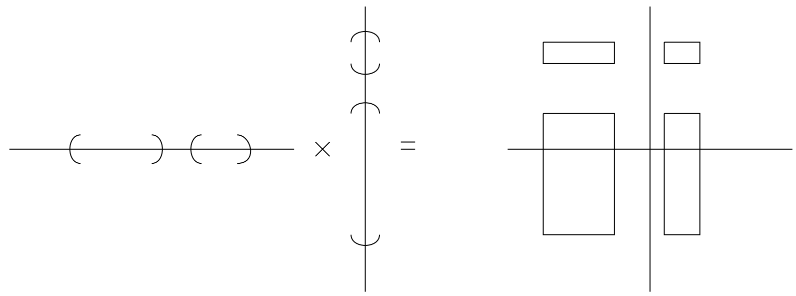


Figure 4.2: Example of the product of two rectilinear embeddings in $\mathbb{D}^1$ which gives an embedding in $\mathbb{D}^2$.

In Figure 4.2 the reader can see an intuitive picture of the product of $\mathbb{E}_k$ operads. Notice that by definition $\times : \mathrm{Disk}_k^{\otimes} \times \mathrm{Disk}_{k'}^{\otimes} \to \mathrm{Disk}_{k+k'}^{\otimes}$ maps inert morphisms to inert morphism and we have a commutative diagram of topological categories

$$\begin{tikzcd}
\mathrm{Disk}_k^{\otimes} \times \mathrm{Disk}_{k'}^{\otimes} \arrow[r] \arrow[d] & \mathrm{Disk}_{k+k'}^{\otimes} \arrow[d] \\
\mathrm{Fin}_* \times \mathrm{Fin}_* \arrow[r, "\wedge"] & \mathrm{Fin}_*
\end{tikzcd}.$$

Taking the homotopy coherent nerve we obtain the desired bifunctor of quasioperads.

Theorem 4.1.8. Let $k, k' \geq 0$. Then the map of quasioperads

$$N(\times) : \mathbb{E}_k \times \mathbb{E}_{k'} \to \mathbb{E}_{k+k'},$$

is a tensor product of quasioperads. In particular, for any symmetric monoidal quasicategory we have an equivalence of quasicategories

$$\mathrm{Alg}_{\mathbb{E}_{k+k'}}(\mathcal{C}^{\otimes}) \cong \mathrm{BiFunc}(\mathbb{E}_k, \mathbb{E}_{k'};\ \mathcal{C}^{\otimes}).$$

Proof. The theorem was first carried in the topological setting by Dunn in [Dun88]. Later it was proved for quasioperads by Lurie in [Lur09a, Theorem 1.2.2.]. $\square$

Definition 4.1.9. Let $\mathcal{C}$ be a quasicategory with finite limits and $(\mathcal{O}^{\otimes}, p)$, $(\mathcal{O}'^{\otimes}, p')$ a pair of quasioperads. A **lax Cartesian bifunctor** is a map of simplicial sets $\pi : \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{C}$ satisfying the following condition:

- Let $C \times D = (C_i, D_j)_{1 \leq i \leq n, 1 \leq j \leq m}$ be an object $\mathcal{O}^{\otimes}_{\langle n \rangle} \times \mathcal{O}'^{\otimes}_{\langle m \rangle} \cong \mathcal{O}^{\times nm}_{\langle 1 \rangle}$. We require that the maps $\pi_{ij} : \pi(C) \to \pi(C_{ij})$ exhibit $\pi(C)$ as an a product $\Pi_{1 \leq i \leq n, 1 \leq j \leq m} \pi(C_{ij})$ in $\mathcal{C}$.

Let $\underline{\mathrm{sSets}}^{\mathrm{BiLax}}(\mathcal{O}^{\otimes}, \mathcal{C})$ be the full quasicategory of $\underline{\mathrm{sSets}}(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}, \mathcal{C})$ spanned by lax Cartesian bifunctors.

Proposition 4.1.10. For every quasicategory $\mathcal{C}$ with finite limits and every pair of quasioperads $\mathcal{O}^{\otimes}$, $\mathcal{O}'^{\otimes}$ there exist an equivalence of quasicategories

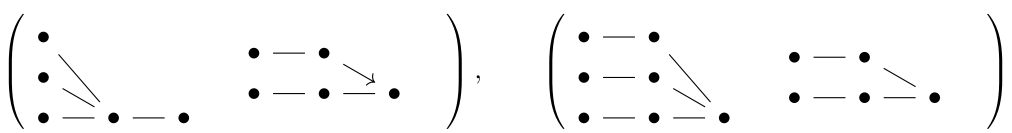
$$\mathrm{BiFunc}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}; \mathcal{C}^{\times}) \cong \underline{\mathrm{sSets}}^{\mathrm{BiLax}}(\mathcal{O}^{\otimes}, \mathcal{C}).$$

Proof. The proof is a slight generalization of [Lur09b, Proposition 2.4.1.7]. Due to its technical nature we will postpone it to Appendix C, where we will discuss Cartesian symmetric monoidal quasicategories. □

Remark 4.1.11. By Theorem 4.1.8 and 4.1.10 to understand an $\mathbb{E}_2$-algebra on a Cartesian monoidal quasicategory $\mathcal{C}$ we need to consider maps of simplicial sets out of $\mathbb{A}\mathrm{ss} \times \mathbb{A}\mathrm{ss}$ that are Lax Cartesian bifunctors. In order give an explicit description of $\mathbb{E}_2$-algebras we first restrict the simplices that we need to consider:

1. It is enough to consider products of trees, since in general a product of coproducts of trees is a coproduct of products of trees.

2. We can restrict to nondegenerate simplices. Since the degeneracy of the product is the product of the degeneracies, we see that nondegerenrate simplices are represented by product of trees that do not have all a horizontal branch in the same place. For example, the product in the left is nondegenerate while the one on the right is degenerate:

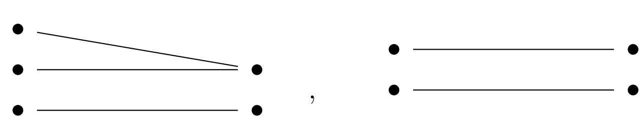


3. Since the operad Ass is generated as an operad by Ass(2) and Ass(0), it follows that every tree appears as the face of a full binary tree, thus we may restrict to products of binary trees.

4. Again, in this thesis we will restrict ourselves to Cartesian monoidal structures. By the property of lax Cartesian bifunctors we see that a permutation of the labeling of the tree is associated to a permutation of the domain of the morphism $\mathcal{C}^{nm} \to \mathcal{C}$. Thus if we fix an order on the product, we may omit the labeling on the trees.

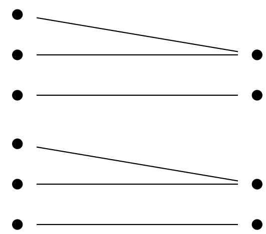
5. Let $\varphi : \mathbb{E}_1 \times \mathbb{E}_1 \to \mathcal{C}^{\times}$ be a Lax Cartesian bifunctor and let C be the image of the unique objects in $(\mathbb{A}\mathrm{ss} \times \mathbb{A}\mathrm{ss})_{\langle 1 \rangle}$. If (α, β) is a pair of trees over $\langle n \rangle \to \langle n' \rangle$ and $\langle m \rangle \to \langle m' \rangle$ respectively, then $\varphi(\alpha, \beta)$ corresponds to a map $C^{nm} \to C^{n'm'}$ (the structure of this maps is determined by the wedge product of finite sets). By our restriction on the trees we will be considering it is enough to understand this map for the following cases:

   - First $\alpha = \mu_{ij}$ is the tree over $\langle n \rangle \to \langle n-1 \rangle$ joining the ith and jth inputs and β is the unit tree. For example,



$\langle 3 \rangle \to \langle 2 \rangle$, $\langle 2 \rangle \to \langle 2 \rangle$ (4.6)

     In this case the map $C^{nm} \to C^{(n-1)m}$ can be described as follows: divide C^{nm} in m blocks of n elements, then take the product of the ith and jth elements in each block to obtain m blocks each with $(n-1)$ elements. Equivalently we are applying the multiplication map $C^n \to C^{n-1}$ simultaneously on the m copies of $(C^n)^m$ to obtain a map $(C^n)^m \to (C^{(n-1)})^m$. For example, for the tree in (4.6) the map is

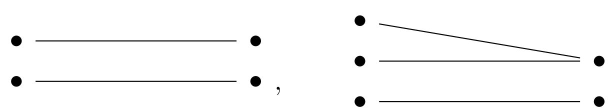

,

$\langle 6\rangle \to \langle 4\rangle$

which on objects looks like

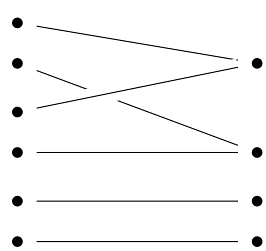
$$C^6 \to C^4$$
$$(X_1, X_2, X_3, X_4, X_5, X_6) \mapsto (X_1 \otimes X_2, X_3, X_4 \otimes X_5, X_6).$$

- Second α is the unit tree and $\beta = \mu_{ij}$ is the tree over $\langle m\rangle \to \langle m-1\rangle$ joining the ith and jth inputs. For example,


(4.7)

$\langle 2\rangle \to \langle 2\rangle$ $\qquad$ $\langle 3\rangle \to \langle 2\rangle$

In this case the map $C^{nm} \to C^{n(m-1)}$ can be described as follows: divide C^{nm} in m blocks of n elements, then multiply componentwise the elements in the ith and jth blocks to obtain $(m-1)$ blocks of n elements. Equivalently, we are considering C^n as an algebra with the pointwise product structure and the map is given by taking the product of the ith and jth components $(C^n)^m \to (C^n)^{(m-1)}$. For the trees in (4.7) the map is


,

$\langle 6\rangle \to \langle 4\rangle$

which on objects looks like

$$C^6 \to C^4$$
$$(X_1, X_2, X_3, X_4, X_5, X_6) \mapsto (X_1 \otimes X_3, X_2 \otimes X_4, X_5, X_6).$$

4.2 Braided Monoidal Categories as $\mathbb{E}_2$-algebras

Our aim in this section is to use Dunn's additivity to prove Theorem 0.0.4: $\mathbb{E}_2$-algebras on $\mathbb{C}\mathrm{at}$ describe and are described by braided monoidal categories. This result was already stated by Lurie in [Lur09a, Example 5.1.2.4.], however here we present a detailed proof which can be generalized to $\mathbb{G}\mathrm{ray}$. For convenience of the reader we will briefly define braided monoidal categories.

Definition 4.2.1. A **strict braided monoidal category** $(C, \otimes, I, \sigma)$ is a monoidal category $(c, \otimes, I)$ together with a natural transformation $\sigma : \otimes \Rightarrow \otimes^{op}$, where $\otimes^{op} = \otimes \circ \tau$ and τ is the flip map, given on components by $\sigma_{X,Y} = X \otimes Y \to Y \otimes X$, such that the following diagrams commutes

$$\begin{tikzcd} X \otimes Y \otimes Z \arrow[rr, "\sigma_{X,Y\otimes Z}"] \arrow[d, "\sigma_{X,Y}\otimes id_Z"'] & & Y \otimes Z \otimes X \\ Y \otimes X \otimes Z \arrow[urr, "id_Y \otimes \sigma_{X,Z}"'] & & \end{tikzcd} \qquad \begin{tikzcd} X \otimes Y \otimes Z \arrow[rr, "\sigma_{X\otimes Y,Z}"] \arrow[d, "id_X\otimes \sigma_{Y,Z}"'] & & Z \otimes X \otimes Y \\ X \otimes Z \otimes Y \arrow[urr, "\sigma_{X,Z} \otimes id_Y"'] & & \end{tikzcd}$$

Remark 4.2.2. One can also consider non strict braided monoidal categories. The definition is similar to the one above, however the braiding is required to satisfy a hexagon diagram. The hexagon arise from adding an associator edge on the vertices of the triangles in Definition 4.2.1. We consider the strict version just for convenience, but remark that the proof of Theorem 0.0.4 may be carried to the non strict setting.

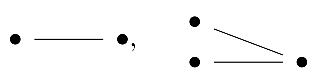
Proof of 0.0.4. The proof has two steps:

1. We use Dunn's additivity to show that an $\mathbb{E}_2$-algebra structure on $\mathbb{C}\mathrm{at}$ determines and is determined by a monoidal category $(\mathcal{C}, \otimes)$ such that $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a monoidal functor (where we take the product monoidal structure $\otimes_2$ on the left). By Dunn's additivity it is enough to consider the images in $\mathbb{C}\mathrm{at}_{(2,1)}$ of the simplices in $\mathbb{A}\mathrm{ss} \times \mathbb{A}\mathrm{ss}$ described in remark 4.1.11.

2. Show that a $\otimes$-monoidal functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ determines and is determined by a braiding in $(C, \otimes)$.

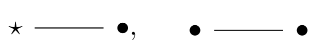
Step 1: Let C be the image of the unique object in $(\mathbb{A}\mathrm{ss} \times \mathbb{A}\mathrm{ss})_{\langle 1 \rangle}$.

1. 1-simplices: The possible products of trees of length 1 are

 $\mapsto \quad C^2 \xrightarrow{\otimes'} C$

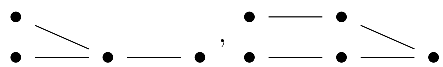
 $\mapsto \quad C^2 \xrightarrow{\otimes} C$

 $\mapsto \quad C^0 \xrightarrow{I'} C$

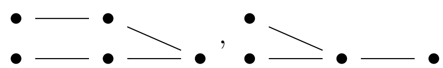
 $\mapsto \quad C^0 \xrightarrow{I} C$

Notice that $\zeta_2 : \langle 2 \rangle \to \langle 1 \rangle$ and $\zeta_1 : \langle 2 \rangle \to \langle 1 \rangle$ satisfy $\zeta_1 \wedge \zeta_2 = \zeta_2 \wedge \zeta_1$, then $\otimes = \otimes'$ and $I = I'$. Thus 1-simplices define a monoidal product and a unit in C.

2. 2-simplices: We now consider trees of length 2 and the images are given by triangles. The (non degenerate) product of trees of length 2 and their images are:



$$\mapsto \quad \begin{tikzcd} & C^2 \arrow[dr, "\otimes"] & \\ C^4 \arrow[ur, "\otimes\times\otimes"] \arrow[rr, "F_4"'] \arrow[rr, phantom, shift left=4ex, "\Downarrow\xi_1"] & & C \end{tikzcd}$$



$$\mapsto \quad \begin{tikzcd} & C^2 \arrow[dr, "\otimes"] & \\ C^4 \arrow[ur, "\otimes_2"] \arrow[rr, "F_4"'] \arrow[rr, phantom, shift left=4ex, "\Downarrow\xi_2"] & & C \end{tikzcd}$$

Here we used the description of the maps determined by a product of trees as in Remark 4.1.11 to describe the maps $C^4 \to C^2$ (see also Example 4.1.3). Since ξ_1, ξ_2 are invertible natural transformations, we can invert one of these morphism to get a natural isomorphism $\varphi := \xi_1^{-1}\xi_2 : \otimes \circ (\otimes_2) \Rightarrow \otimes \circ (\otimes \times \otimes)$. Which on components is given by:

$$\varphi_{(X_1,X_2|X_3,X_4)} : X_1 \otimes X_2 \otimes X_3 \otimes X_4 \to X_1 \otimes X_3 \otimes X_2 \otimes X_4$$

Product of trees with 0-ary operations give compatibility between φ and the unitors of $\otimes$ and $\otimes_2$. Keeping track of these structures is cumbersome and will not bring any insight on the structure of arguments in the future, thus we omit them. This is equivalent to considering the unitors to be the identity (which is not harmful since we may always strictify units in monoidal categories).

3. 3-simplices: We now consider products of trees of length 3 and the images are given by filled tetahedra. The possible trees of length 3 and their images are completely described in Appendix B. In this case there are 12 triangles, and all of them can be arranged into the diagrams

$$\begin{tikzcd} & & F_3\circ(\otimes\times\otimes\times\otimes) \arrow[dll, "\xi_1^{-1}\xi_2\times id^2"'] \arrow[d] \arrow[drr, "id^2\times\xi_1^{-1}\xi_2"] & & \\ \otimes\circ(\otimes\times\otimes)\circ(\otimes_2\times id^2) \arrow[rr] \arrow[drr, "\xi_1^{-1}\xi_2 * id_{\otimes_2\times id^2}"'] & & F_6 & & \otimes\circ(\otimes\times\otimes)\circ(id^2\times\otimes_2) \arrow[ll] \arrow[dll, "\xi_1^{-1}\xi_2 * id_{id^2\times\otimes_2}"] \\ & & \otimes\circ\otimes_2\circ(\otimes_2\times id^2)=\otimes\circ\otimes_2\circ(id^2\times\otimes_2) \arrow[u] & & \end{tikzcd} \tag{4.8}$$

$$
\begin{tikzcd}
 & & F_3\circ(\otimes_3) \arrow[ddll] \arrow[dd] \arrow[ddrr] & & \\
 & \xi_1^{-1}\xi_2 * \otimes_{3,6} \arrow[dr] & & \xi_1^{-1}\xi_2 * \otimes_{1,4} \arrow[dl] & \\
\otimes\circ\otimes_2\circ(\otimes\times id)^2 \arrow[ddrr] & & F_6 & & \otimes\circ\otimes_2\circ(id\times\otimes)^2 \arrow[ddll] \\
 & \xi_1^{-1}\xi_2 * id_{\otimes_2\times id^2} \arrow[ur] & & \xi_1^{-1}\xi_2 * id_{id^2\times\otimes_2} \arrow[ul] & \\
 & & \otimes\circ(\otimes\times\otimes)\circ(id\times\otimes)=\otimes\circ(\otimes\times\otimes)\circ(\otimes\times id) \arrow[uu] & &
\end{tikzcd}
\tag{4.9}
$$

Where in the last diagram $\otimes_{i,j}$ means the tensor product of the objects in i, j position. To get an idea of the significance of these diagrams it is useful to consider them in components. For example the Diagram 4.8 in components is given by the commutative diagrams

$$
\begin{tikzcd}
 & X_1X_2X_3X_4X_5X_6 \arrow[dl, "\varphi_{(X_1,X_2|X_3,X_4)}\otimes id_{X_5\otimes X_6}"'] \arrow[dr, "id_{X_1\otimes X_2}\otimes\varphi_{(X_3,X_4|X_5,X_6)}"] & \\
X_1X_3X_2X_4X_5X_6 \arrow[dr, "\varphi_{(X_1,X_2|X_3\otimes X_5,X_4\otimes X_6)}"'] & & X_1X_2X_3X_5X_4X_6 \arrow[dl, "\varphi_{(X_1\otimes X_3,X_2\otimes X_4|X_5,X_6)}"] \\
 & X_1X_3X_5X_2X_4X_6 &
\end{tikzcd}
$$

Here we omitted the tensor product $\otimes$ and use the notation X_1X_2 instead of $X_1 \otimes X_2$. This is exactly the data of exhibiting $\otimes : C \times C \to C$ as a $\otimes$-monoidal functor. On the other hand, the trees with 0-ary operations give the conditions

$$\varphi_{(I,I|X,Y)} = I = \varphi_{(X,Y|I,I)}.$$

We remark that in this case the Diagram 4.9 is consequence of Diagram 4.8 and the unit condition, thus it not relevant (this will not be true in the case of **G**ray).

Before going into step 2, we record some consequences of the strict unitality constraint:

1. Taking
$$X_1 = X, \quad X_2 = Y, \quad X_3 = Z, \quad X_4 = I, \quad X_5 = I, \quad X_6 = I,$$
we see that $\varphi(X, Y|Z, I) \otimes id_W = \varphi(X, Y|Z, W)$. Similarly, $\varphi(X, Y|Z, W) = id_X \otimes \varphi(I, X|Y, Z)$

2. Taking
$$X_1 = I, \quad X_2 = X, \quad X_3 = I, \quad X_4 = I, \quad X_5 = I, \quad X_6 = I$$
and using the the property 1 above, we see that $\varphi(X, Y|I, Z) = id_{X\otimes Y\otimes Z}$ and similarly $\varphi(X, I|Y, Z) = id_{X\otimes Y\otimes Z}$.

In the case we did not have strict units similar relations hold replacing identities by unitors.

Step 2: Suppose there is a $\otimes$-monoidal functor functor $(\otimes, \varphi)$, then we claim $\sigma_{X,Y} := \varphi_{(I,X|Y,I)}$ defines a braiding on $(\mathcal{C}, \otimes)$. Clearly σ defines a natural transformation, naturality following from that of φ. The braiding conditions follow from the coherence condition of the monoidal functor, indeed considering

$$X_1 = I, \quad X_2 = X, \quad X_3 = Y, \quad X_4 = I, \quad X_5 = Z, \quad X_6 = I,$$

the coherence condition for a monoidal functor implies

$$\begin{tikzcd}[column sep=small]
 & X \otimes Y \otimes Z \arrow[dl, "\varphi(I,X|Y,I)\otimes id_Z=\sigma_{X,Y}\otimes id_Z"'] \arrow[dr, "id_X\otimes\varphi_{(Y,I|Z,I)}=id_{X\otimes Y\otimes Z}"] & \\
Y \otimes X \otimes Z \arrow[dr, "\varphi_{(Y,X|Z,I)}=id_Y\otimes\sigma_{X,Z}"'] & & X \otimes Y \otimes Z \arrow[dl, "\varphi_{(I,X|Y\otimes Z,I)}=\sigma_{X,Y\otimes Z}"] \\
 & Y \otimes Z \otimes X &
\end{tikzcd}$$

which is exactly one of the braiding triangles. Similarly taking

$$X_1 = I, \quad X_2 = X, \quad X_3 = I, \quad X_4 = Y, \quad X_5 = Z, \quad X_6 = I,$$

one obtains the other braiding triangle.

Now assume there is a braiding $\sigma_{X,Y} : X \otimes Y \to Y \otimes X$ on $(\mathcal{C}, \otimes)$, then we will prove that $\varphi_{(X_1,X_2|X_3,X_4)} = id_{X_1} \otimes \sigma_{X_1,X_2} \otimes id_{X_4}$ endows $\otimes$ with the structure of a $\otimes$-monoidal functor. Indeed, we can decompose the commutative quadrilateral into commutative triangles.

$$\begin{tikzcd}
 & X_1X_2X_3X_4X_5X_6 \arrow[dl, "\sigma_{X_2,X_3}\otimes id_{X_4X_5}"'] \arrow[dr, "id_{X_2X_3}\otimes\sigma_{X_4,X_5}"] & \\
X_1X_3X_2X_4X_5X_6 \arrow[r, "id_{X_3X_2}\otimes\sigma_{X_4,X_5}"] \arrow[dr, "id_{X_3}\otimes\sigma_{X_2X_4,X_5}"'] & X_1X_3X_2X_5X_4X_6 \arrow[d, "id_{X_3}\otimes\sigma_{X_2,X_5}\otimes id_{X_4}"] & X_1X_2X_3X_5X_4X_6 \arrow[l, "\sigma_{X_2,X_3}\otimes id_{X_5X_4}"'] \arrow[dl, "\sigma_{X_2,X_3}\otimes id_{X_5X_4}"] \\
 & X_1X_3X_5X_2X_4X_6 &
\end{tikzcd}$$

□

4.3 The $\mathbb{E}_n$-operad Tower and the Stabilization Hypothesis

We come back to study of the operads $\mathbb{E}_n$, for this it is useful to have a nice topological model for its mapping spaces. A useful model for these spaces is given by configurations spaces. We will see that many properties of the $\mathbb{E}_n$ operads can be obtained from the topology of configuration spaces.

Definition 4.3.1. The **ordered configuration space of n distinct points in** $\mathbb{R}^d$, denoted $\mathrm{Conf}_n(\mathbb{R}^d)$ is the subspace of $(\mathbb{R}^d)^{\times n}$ given by

$$\mathrm{Conf}_n(\mathbb{R}^d) = \{\ (x_1, \ldots, x_n) \in (\mathbb{R}^d)^{\times n} \mid x_i \neq x_j \text{ if } i \neq j\}.$$

Proposition 4.3.2. The space $\mathrm{Disk}_d(n) = \mathrm{Rect}(\coprod_d \mathbb{D}^n, \mathbb{D}^n)$ is homotopy equivalent to $\mathrm{Conf}_n(\mathbb{R})$.

Proof. To each rectilinear embedding $f(x) = \lambda x + \mu$ we can associate its center $\mu \in \mathbb{R}^d$, this defines a projection $\mathrm{Rect}(\mathbb{D}, \mathbb{D}^d) \to \mathrm{Conf}_1(\mathbb{R}^d)$. Since elements in $\mathrm{Disk}_d(n)$ are disjoint rectilinear embeddings, in particular have different centers, we can extend the previous map to a fibration $\mathrm{Disk}_d(n) \to \mathrm{Conf}_n(\mathbb{R}^d)$. This projection defines a fiber bundle whose fibers are given by $(\mathbb{R}^+)^n$, the set of possible radii. Since the fibers are contractible spaces, the desired statement follows. $\square$

Proposition 4.3.3. The configuration space $\mathrm{Conf}_n(\mathbb{R})$ is $(d-2)$-connected, i.e. $\pi_m(\mathrm{Conf}_n(\mathbb{R})) = 0$ for $1 < m \leq d-2$.

Proof. This a consequence of the following 2 facts:

1. The configuration space $\mathrm{Conf}_2(\mathbb{R})$ is homotopic to the sphere S^{d-1}: Indeed any two different points are determined by the relative position between them. Explicitly we have a map an homeomorphism

$$\mathrm{Conf}_2(\mathbb{R}) \to \mathbb{R}^d \times S^{d-1} \times \mathbb{R}^+, \qquad (x_1, x_2) \mapsto (x_1, |x_1 - x_2|, \frac{x_1 - x_2}{|x_1 - x_2|}).$$

   Since both $\mathbb{R}^+$ and $\mathbb{R}^d$ are contractible, then the claim follows.

2. Let $\pi_i : \mathrm{Conf}_n(\mathbb{R}^d) \to \mathrm{Conf}_{n-1}(\mathbb{R}^d)$ be the projection skipping the ith component $(x_1, \dots, x_n) \mapsto (x_1, \dots, \hat{x}_i, \dots, x_n)$. This projection determines a fiber bundle with fiber $\mathbb{R}^d \setminus \{$(n-1) points$\}$. Moreover since $\mathbb{R}^d \setminus \{$(n-1) points$\}$ retracts to $\bigvee_{n-1} S^{d-1}$ we have a homotopy fiber sequence

$$\bigvee_{n-1} S^{d-1} \to \mathrm{Conf}_n(\mathbb{R}^d) \xrightarrow{p_n} \mathrm{Conf}_{n-1}(\mathbb{R}^d).$$

   For a proof of this the reader may consult [FN62, Theorem 1] .

Using the long exact sequence in homotopy of the fibrations in 2 we obtain inductively that $\pi_m(\mathrm{Conf}_n(\mathbb{R})) = 0$ for $1 < m \leq d-2$, where the base case is given by the fact that the sphere S^{d-1} is $(d-2)$-connected. $\square$

Corollary 4.3.4. The inclusion $\mathbb{R}^d \to \mathbb{R}^{d+1}$ into the first d coordinates determines a fibration of topological spaces $\mathrm{Conf}_n(\mathbb{R}^d) \to \mathrm{Conf}_n(\mathbb{R}^{d+1})$. Thus there is a (filtered) sequence of topological spaces

$$\mathrm{Conf}_n(\mathbb{R}^0) \longrightarrow \mathrm{Conf}_n(\mathbb{R}^1) \longrightarrow \mathrm{Conf}_n(\mathbb{R}^2) \longrightarrow \cdots . \tag{4.10}$$

Let $\mathrm{Conf}_n(\mathbb{R}^\infty)$ denote the limit of the sequence (4.10), then $\mathrm{Conf}_n(\mathbb{R}^\infty)$ is contractible.

Proof. By [Hat05, Proposition 4.67]), the natural map

$$\pi_i(\lim_{d \longrightarrow \infty} \mathrm{Conf}_n(\mathbb{R}^d)) \to \lim_{d \longrightarrow \infty} \pi_i(\mathrm{Conf}_n(\mathbb{R}^d)),$$

for some fixed $i \in \mathbb{N}$, is injective if the maps $\pi_{i+1}(\mathrm{Conf}_n(\mathbb{R}^d)) \to \pi_{i+1}\mathrm{Conf}_n(\mathbb{R}^{d-1})$ are surjective for high enough d. Since $\pi_{i+1}\mathrm{Conf}_n(\mathbb{R}^{d-1})$ vanishes for high enough d, we conclude that in the limit all homotopy groups of $\mathrm{Conf}_n(\mathbb{R}^\infty)$ vanish, i.e $\mathrm{Conf}_n(\mathbb{R}^\infty)$ is contractible. $\square$

Proposition 4.3.5. By taking the product with $\mathbb{D}^1$ one can define a map between rectilinear embedding

$$(f : \coprod \mathbb{D}^k \to \mathbb{D}^k) \mapsto (f \times id_{\mathbb{D}^1} : \coprod \mathbb{D}^k \times \mathbb{D}^1 \to \mathbb{D}^k \times \mathbb{D}^1),$$

which determines an inclusion of quasioperads $\mathbb{E}_n \to \mathbb{E}_{n+1}$. Denote by $\mathbb{E}_\infty$ the limit of the sequence of quasioperads

$$\mathbb{E}_0 \longrightarrow \mathbb{E}_1 \longrightarrow \mathbb{E}_2 \longrightarrow \cdots .$$

Then the canonical map $p : \mathbb{E}_\infty \to \mathrm{N}(\mathrm{Fin}_*)$ determines an equivalence of quasioperads, where we consider $(N(\mathrm{Fin}_*), id)$ as a quasioperad via the identity morphism.

Proof. Notice that for every $k \geq 0$ the quasioperad $\mathbb{E}_k$ and $N(\mathrm{Fin}_*)$ have the same objects. Passing to the limits we conclude that the quasicategories $\mathbb{E}_\infty$ and $N(\mathrm{Fin}_*)$ have the same objects. Thus to prove that the map $p : \mathbb{E}_\infty \to N(\mathrm{Fin}_*)$ determines an equivalence of quasicategories is enough to prove that the mapping spaces are homotopic. Indeed, by Proposition 4.3.2 we have an homotopy equivalence

$$\begin{aligned}\mathbb{E}_\infty(\langle m\rangle, \langle n\rangle) &= \lim_{n\to\infty} \coprod_{\phi:\langle m\rangle\to\langle n\rangle} \prod_{1\leq j\leq n} \mathrm{Rect}\Big(\coprod_{\phi^{-1}(j)} \mathbb{D}^k, \mathbb{D}^k\Big) \\ &\sim \lim_{n\to\infty} \coprod_{\phi:\langle m\rangle\to\langle n\rangle} \prod_{1\leq j\leq n} \mathrm{Conf}_{\phi^{-1}(j)}(\mathbb{R}^k)\end{aligned}$$

Moreover, since filtered colimits commute with finite products and coproducts we have an homotopy equivalence

$$\begin{aligned}\mathbb{E}_\infty(\langle m\rangle, \langle n\rangle) &= \lim_{n\to\infty} \coprod_{\phi:\langle m\rangle\to\langle n\rangle} \prod_{1\leq j\leq n} \mathrm{Conf}_{\phi^{-1}(j)}(\mathbb{R}^k) \\ &= \coprod_{\phi:\langle m\rangle\to\langle n\rangle} \prod_{1\leq j\leq n} \mathrm{Conf}_{\phi^{-1}(j)}(\mathbb{R}^\infty) \\ &\sim \coprod_{\phi:\langle m\rangle\to\langle n\rangle} * = \mathrm{Fin}_*(\langle m\rangle, \langle n\rangle),\end{aligned}$$

where in the last line we made use of Corollary 4.3.4 □

From the proof of Proposition 4.3.5 we see that the quasioperad $\mathbb{E}_\infty \cong N(\mathrm{Fin}_*)$ has contractible spaces of operations. Thus there exist, up to homotopy, only one way of multiplying n element without considering their order. That is, algebras over the quasioperad $N(\mathrm{Fin}_*)$ are **homotopy commutative algebras**. Moreover, Proposition 4.3.5 gives the intuition for $\mathbb{E}_n$-algebras described in the introduction: $\mathbb{E}_1$-algebras describe homotopy associative algebras, on the other extreme $\mathbb{E}_\infty$-algebras describe homotopy commutative algebras, and for for $1 < n < \infty$, $\mathbb{E}_n$-algebras describe algebras with intermediate levels of homotopy commutativity. Before ending this section we will see that if the target symmetric monoidal category is n-coskeletal, then the algebras $\mathbb{E}_{n+1}$ are already homotopy commutative. The previous important phenomenon is sometimes referred as the **stabilization hypothesis** and was originally proposed by Baez and Dolan [BN96].

Theorem 4.3.6. Let $\mathcal{C}$ be a quasicategory that admits finite products and is equivalent to an $(n, 1)$-category. Then for $k > n$ the map $\mathbb{E}_k \to \mathbb{E}_\infty$ induces an equivalence of quasicategories

$$\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}^\times) \xrightarrow{\sim} \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}^\times).$$

Proof. We will start with an argument that works for a general quasioperad $\mathcal{O}^\otimes$. Recall that by definition of the Cartesian symmetric monoidal structure we have that the $\mathrm{Alg}_{\mathcal{O}^\otimes}(\mathcal{C}^\times) \cong \underline{\mathrm{sSets}}^{\mathrm{Lax}}(\mathcal{O}^\otimes, \mathcal{C})$.Morover, since the category $\mathcal{C}$ is $(n+1)$-coskeletal we have

$$\underline{\mathrm{sSets}}(\mathcal{O}^\otimes, \mathcal{C}) = \underline{\mathrm{sSets}}(h_{(n+1)}\mathcal{O}^\otimes, \mathcal{C}) \quad \text{and} \quad \underline{\mathrm{sSets}}^{\mathrm{Lax}}(\mathcal{O}, \mathcal{C}) = \underline{\mathrm{sSets}}^{\mathrm{Lax}}(h_{n+1}\mathcal{O}^\otimes, \mathcal{C}).$$

Where in the second equality we used the fact that the condition of being a lax Cartesian structure is given on objects, and the objects of $\mathcal{O}^{\otimes}$ are the same objects of $h_{n+1}\mathcal{O}^{\otimes}$.

We come back to the specific case of $\mathbb{E}_k$ operads. By 2.4.10 the mapping spaces of $h_{n+1}\mathbb{E}_k$ are $(n+1)$-truncated, and by 4.3.3 they are $(n+2)$-connected. Therefore the homotopy groups of all mapping spaces of $\mathbb{E}_k$ vanish. By the same arguments as in 4.3.5 there is an equivalence of quasicategories $h_n\mathbb{E}_k \cong \mathbb{E}_\infty$, and we conclude

$$\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}^{\times}) \cong \underline{\mathrm{Sets}}^{\mathrm{Lax}}(h_n\mathcal{O}^{\otimes}, \mathcal{C}) \cong \underline{\mathrm{sSets}}^{\mathrm{Lax}}(\mathbb{E}_\infty, \mathcal{C}) = \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}^{\times}).$$

□

Remark 4.3.7. From the discussions above we can give an informal slogan summarizing Dunn's additivity and the stabilization hypothesis: "A compatible homotopy associative algebra structure on the higher category of $\mathbb{E}_n$-algebras adds an additional layer of commutativity, moreover after adding enough layers of commutativity we obtain a homotopy commutative algebra". In the case of monoids, due to the 2-coskeletality of $N(\mathrm{Sets})$ we have a $\mathbb{E}_2$-algebra is already a commutative monoid. This explains the classical Eckamnn-Hilton argument from a higher category perspective. Informally we may view the iterated use of Dunn's additivity to describe higher $\mathbb{E}_n$-algebras as a higher categorical analogue of the Eckmann-Hiltonn argument.

4.4 Description of $\mathbb{E}_1$-algebras on $\mathbb{G}\mathrm{ray}^{\times}_{(3,1)}$: Monoidal 2-Categories

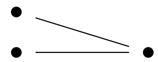
At the end of chapter 3 we showed that an $\mathbb{E}_1$-algebra on $\mathbb{C}\mathrm{at}^{\times}$ was the same as a monoidal category. Now using the same ideas we will prove the generalization to 2-categories, namely Theorem 0.0.3 stated in the introduction: An $\mathbb{E}_1$-algebra on $\mathbb{G}\mathrm{ray}^{\times}_{(3,1)}$ describes and is described by a monoidal 2-category. We have not defined monoidal 2-categories and we choose not to do so here, since its long definition will arise naturally from the description presented in the proof. Therefore we opted for an *on the march* approach were we will sketch the proof of 0.0.6, and in doing so we hope the reader may catch the main concepts behind the definition of a monoidal 2-category. For a complete definition of a monoidal bicategory we refer the reader to [SP09, Appendix C].

Proof of theorem 0.0.3: By proposition 4.1.1 an $\mathbb{E}_1$-algebra is the same as an $\mathbb{A}\mathrm{ss}$-algebra, thus we will follow the same argument as in the proof of theorem 0.0.3. In short the idea is to study the images for most basic n-simplices of the quasioperad $\mathbb{A}\mathrm{ss}$, which are given by binary tress of length n, into the n-simplices of $\mathbb{G}\mathrm{ray}_{(3,1)}$, which are explicitly described in Appendix A. Moreover, 4-coskeletality of $\mathbb{G}\mathrm{ray}_{(3,1)}$ imply that we should only consider n-simplices for $n \leq 4$. Here we will just give an idea how the images of the simplices can be arranged into the data and coherence conditions for a monoidal 2-category. The complete description of the data can be found in Appendix B.

Let $\mathcal{B} := \phi(\bullet)$ be the 2-category determined by the image of the only color in Ass.

1. 1-simplices: We consider trees of length 1. The images for these trees determines 2-functors $\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and $I : \mathcal{B}^0 \to \mathcal{B}$, which gives a monoidal product and unit. Here $\mathcal{B}^0$ is the

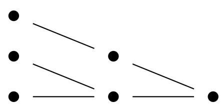
2-category with one object and one morphism.



(••)

$$\mapsto \qquad \mathcal{B}^2 \xrightarrow{\otimes} \mathcal{B}$$

2. 2-simplices: The images of trees of length 2 are given by filled triangles, just as in the proof of theorem 0.0.3.

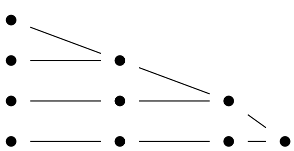


(•(••))

$$\mapsto \qquad \begin{tikzcd} & C^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\alpha_1"] & \\ C^3 \arrow[ur, "\otimes\times\otimes"] \arrow[rr, "F_3"'] & {} & C \end{tikzcd}$$

Here $F_3 : C^3 \to C$ denotes the image of the unique morphism $\langle n \rangle \to \langle 1 \rangle$ in Ass. Similarly we will define $F_n : C^n \to C$ for higher n. Using the 2-cells, that is pseudonatural transformations, and the adjoint equivalence data we can build an associator pseudonatural transformation $\alpha = \alpha_2 \bullet \alpha_1 : \otimes \circ (\otimes \times id) \Rightarrow \otimes \circ (id \times \otimes)$.

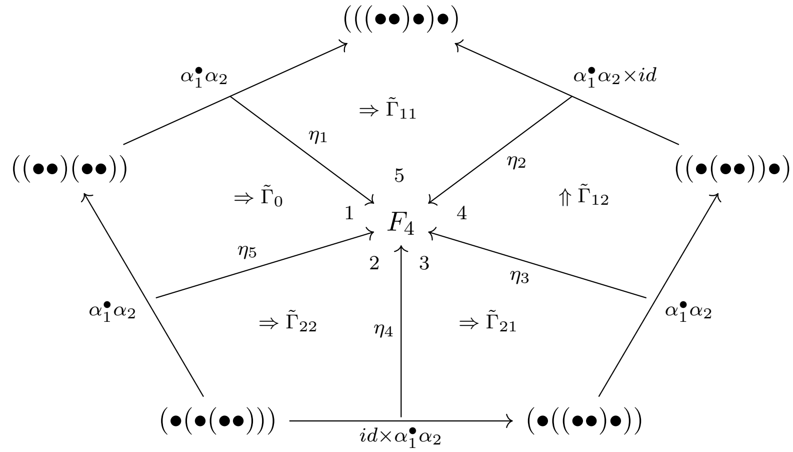
3. 3-simplices: The images of trees of length 3 are given by filled tetrahedra, these are given by 3-cells between pasting diagrams. An example for such a tetaherdra is:



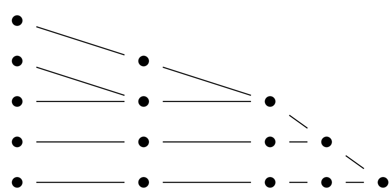
(((••)•)•)

$$\mapsto \qquad \begin{tikzcd} & C^3 \arrow[dr, "\otimes\times id"] \arrow[dd, "F_3"] & \\ C^4 \arrow[ur, "\otimes\times id\times id"] \arrow[dr, "F_4"'] \arrow[r, phantom, "\Leftarrow\eta_2"] & {} \arrow[r, phantom, "\Leftarrow\alpha_1"] & C^2 \arrow[dl, "\otimes"] \\ & C & \end{tikzcd} \quad \Rrightarrow\Gamma_{11} \quad \begin{tikzcd} & C^3 \arrow[dr, "\otimes\times id"] \arrow[d, phantom, "\Downarrow\alpha_1\times id"] & \\ C^4 \arrow[ur, "\otimes\times id\times id"] \arrow[dr, "F_4"'] \arrow[rr, "F_3\times id"] & {} \arrow[d, phantom, "\Downarrow\eta_2"] & C^2 \arrow[dl, "\otimes"] \\ & C & \end{tikzcd}$$

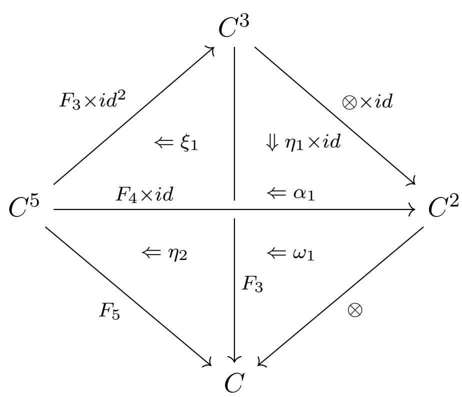
As in the proof of 0.0.3, we can reduce the dimensionality of these pictures by considering them as 2-cells in the category $\mathbb{G}\mathrm{ray}(B^4, \mathcal{B})$. Using this reduction in dimension we can arrange the above tetrahedra, which now become squares, into a pentagon diagram. Using the adjuntion data of the associators we can build modifications $\tilde{\Gamma}$'s from the Γ's. We can compose all the $\tilde{\Gamma}$'s into a 3-cell, that is a modification, between the associators in the exterior of the pentagon. This modification is called the **pentogonator**.



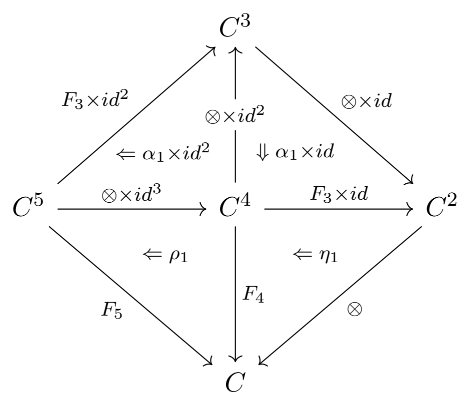
4. 4-simplices: The description of the images of trees of length 4 is given in Appendix B. We will give the full description of one of them. The first bit of data is a *oriented* octahedron, for example for the following length 5 tree the octahedron is given by:



$(((\bullet(\bullet\bullet))\bullet)\bullet)$



Front



Back

The second bit of data for the 4-simplex is a 3-cell between three composable 1-cells in the ocatehedron, there are 5 of them, each given by skipping one of the vertices. We list them here:

$$
\begin{tikzcd}
& C^3 \arrow[dr, "\otimes\times id"] \arrow[dd, "F_3"] & \\
C^4 \arrow[ur, "\otimes\times id\times id"] \arrow[dr, "F_4"'] \arrow[r, phantom, "\Leftarrow \delta_1"] & {} & C^2 \arrow[dl, "\otimes"] \arrow[l, phantom, "\Leftarrow \alpha_1"'] \\
& C &
\end{tikzcd}
\Rrightarrow \Gamma_{11}
\begin{tikzcd}
& C^3 \arrow[dr, "\otimes\times id"] \arrow[d, phantom, "\Downarrow \alpha_1\times id"] & \\
C^4 \arrow[ur, "\otimes\times id\times id"] \arrow[rr, "F_3\times id" description] \arrow[dr, "F_4"'] & {} \arrow[d, phantom, "\Downarrow \eta_1"] & C^2 \arrow[dl, "\otimes"] \\
& C &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& C^3 \arrow[dr, "\otimes\times id"] \arrow[dd, "F_3"] & \\
C^5 \arrow[ur, "F_3\times id^2"] \arrow[dr, "F_5"'] \arrow[r, phantom, "\Leftarrow \xi_1"] & {} & C^2 \arrow[dl, "\otimes"] \arrow[l, phantom, "\Leftarrow \alpha_1"'] \\
& C &
\end{tikzcd}
\Rrightarrow \Theta_{111}
\begin{tikzcd}
& C^3 \arrow[dr, "\otimes\times id"] \arrow[d, phantom, "\Downarrow \eta_1\times id"] & \\
C^5 \arrow[ur, "F_3\times id^2"] \arrow[rr, "F_3\times id" description] \arrow[dr, "F_5"'] & {} \arrow[d, phantom, "\Downarrow \omega_1"] & C^2 \arrow[dl, "\otimes"] \\
& C &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& C^4 \arrow[dr, "F_3\times id"] \arrow[dd, "F_4"] & \\
C^5 \arrow[ur, "\otimes\times id^3"] \arrow[dr, "F_5"'] \arrow[r, phantom, "\Leftarrow \rho_1"] & {} & C^2 \arrow[dl, "F_3"] \arrow[l, phantom, "\Leftarrow \eta_1"'] \\
& C &
\end{tikzcd}
\Rrightarrow \Theta_{112}
\begin{tikzcd}
& C^4 \arrow[dr, "F_3\times id"] \arrow[d, phantom, "\Downarrow \delta_1\times id"] & \\
C^5 \arrow[ur, "\otimes\times id^3"] \arrow[rr, "F_4\times id" description] \arrow[dr, "F_5"'] & {} \arrow[d, phantom, "\Downarrow \omega_1"] & C^2 \arrow[dl, "F_3"] \\
& C &
\end{tikzcd}
$$

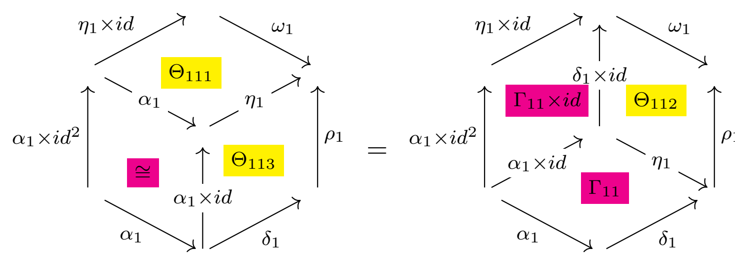
$$
\begin{tikzcd}
& C^4 \arrow[dr, "\otimes\times id"] \arrow[dd, "F_4"] & \\
C^5 \arrow[ur, "\otimes\times id^3"] \arrow[dr, "F_5"'] \arrow[r, phantom, "\Leftarrow \rho_1"] & {} & C^3 \arrow[dl, "F_3"] \arrow[l, phantom, "\Leftarrow \delta_1"'] \\
& C &
\end{tikzcd}
\Rrightarrow \Theta_{113}
\begin{tikzcd}
& C^4 \arrow[dr, "\otimes\times id"] \arrow[d, phantom, "\Downarrow \alpha_1\times id"] & \\
C^5 \arrow[ur, "\otimes\times id^3"] \arrow[rr, "F_3\times id^2" description] \arrow[dr, "F_5"'] & {} \arrow[d, phantom, "\Downarrow \eta_1"] & C^3 \arrow[dl, "F_3"] \\
& C &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& C^4 \arrow[dr, "\otimes\times id^2"] \arrow[dd, "F_3\times id"] & \\
C^5 \arrow[ur, "\otimes\times id^3"] \arrow[dr, "F_4\times id"'] \arrow[r, phantom, "\Leftarrow \delta_1\times id"] & {} & C^3 \arrow[dl, "\otimes\times id"] \arrow[l, phantom, "\Leftarrow \alpha_1\times id"'] \\
& C^2 &
\end{tikzcd}
\Rrightarrow \Gamma_{11}\times id
\begin{tikzcd}
& C^4 \arrow[dr, "\otimes\times id^2"] \arrow[d, phantom, "\Downarrow \alpha_1\times id^2"] & \\
C^5 \arrow[ur, "\otimes\times id^3"] \arrow[rr, "F_3\times id^2" description] \arrow[dr, "F_4\times id"'] & {} \arrow[d, phantom, "\Downarrow \eta_1\times id"] & C^3 \arrow[dl, "\otimes\times id"] \\
& C^2 &
\end{tikzcd}
$$

We can consider each of the 3-cells above as a map between 2-cells in $Gray_{(3,1)}(\mathcal{B}^5, \mathcal{B})$ given by the pasting diagrams, for example the last 3-cell above can be considered a diagram

$$
\begin{tikzcd}
{} \arrow[r, "\alpha_1\times id"] \arrow[d, "\alpha_1\times id^2"'] \arrow[dr, phantom, "\Downarrow \Gamma_{11}\times id"] & {} \arrow[d, "\delta_1\times id"] \\
{} \arrow[r, "\eta_1\times id"'] & {}
\end{tikzcd}
$$

We omitted many of the data of the diagram inside $Gray_{(3,1)}(\mathcal{B}^5, \mathcal{B})$, for example we do not explicitly write the sources and targets of the 1-cells because they can be read from the pasting diagrams. The last piece of data for the 4-simplex is an equality between pasting diagrams in $Gray_{(3,1)}(\mathcal{B}^5, \mathcal{B})$ using the previous 2-cells, which should be considered as cube.



We have colored the faces with two colors: the yellow color represents *interior* faces and the purple color *exterior* faces, the names will become clear soon. Similar to how we arrange the associators into a pentagon we can arrange the pentagonators into a 3-dimensional polytope. The polytope is called the **5-Associahedron** and is has 9 faces from which 6 are pentagons and the other 3 are diamonds, and 21 vertices corresponding to all the possible bracketing of 5 letters. In Figure 4.3 we see how one of the cube of the example above fits inside the Associahedron, here the faces the exterior faces make the exterior of the polytope while the interior faces are inside. In Figure 4.4 we see the full decomposition of the polytope into the 21 cubes given by all the length 5 trees, properly indexed by the bracketing they determine. All the exterior faces are arranged in a way such that they describe pentagonators, and the diamonds are build from commutative rectangles. Moreover the interior faces match each other, implying the exterior of the polytope commutes. Here we are using Lemma 2.2.9 to ignore the direction of the pseudonatural transformations defining the edges of the polytope. In the Figure 4.4 the exterior faces belonging to the same cube are joined by a red point with three markers, moreover the blue cubes are associated to the pentogonator $id \times \pi$, while the purple cubes are associated to $\pi \times id$. The pink cubes are not associated to any particular pentagonator, instead they are pieces of the remaining pentagonators; which are of the form $\pi \circ \otimes_{i,j}$, where $\otimes_{i,j}$ is multiplying the i, j component. The full verification of this facts are given in Appendix B, however we hope the picture is enough to give the reader enough intuition for the proof.

In summary the images of all the trees of length $n \leq 4$ define the following data: an associator, a pentagonator, and coherence between pentagonators. One can use similar arguments to obtain unitors, higher unitors and coherence between them. Without going into full detail, the definition of a monoidal 2-category is given by the previous data and coherence conditions, thus we sketched a proof that a $\mathbb{E}_1$-algebra on $\mathbb{G}\mathrm{ray}_{(3,1)})$ determines a monoidal 2-category. On the other hand given a monoidal 2-category one can describe a $\mathbb{E}_1$-algebra where most of the images of the trees are degenerate cells (i,e. given by identity n-cells) and just some small set (for example just the γ_{11}'s) of it given by the 2-monoidal category data. $\square$

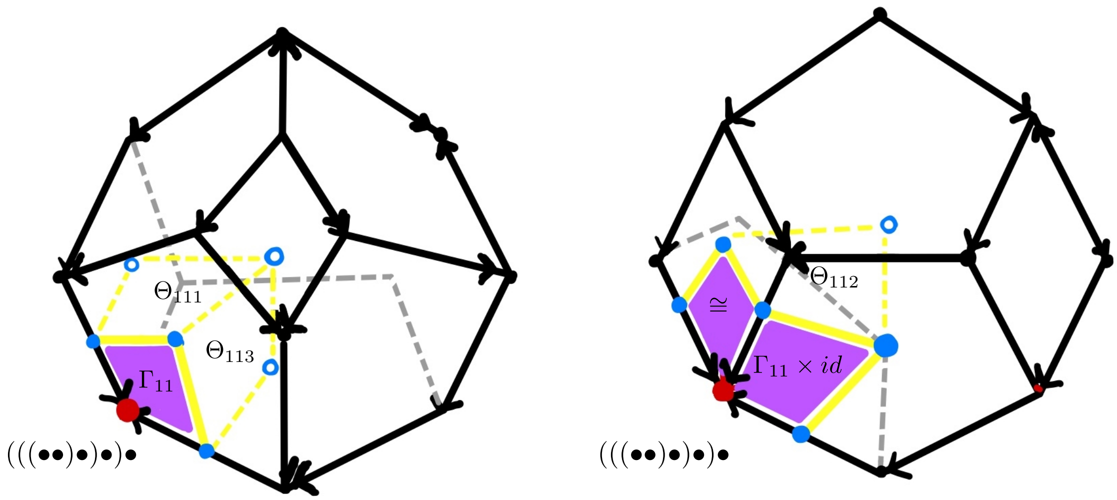


Figure 4.3: Example of a cube determined by a length 4 bracketing tree inside the Associahedron.

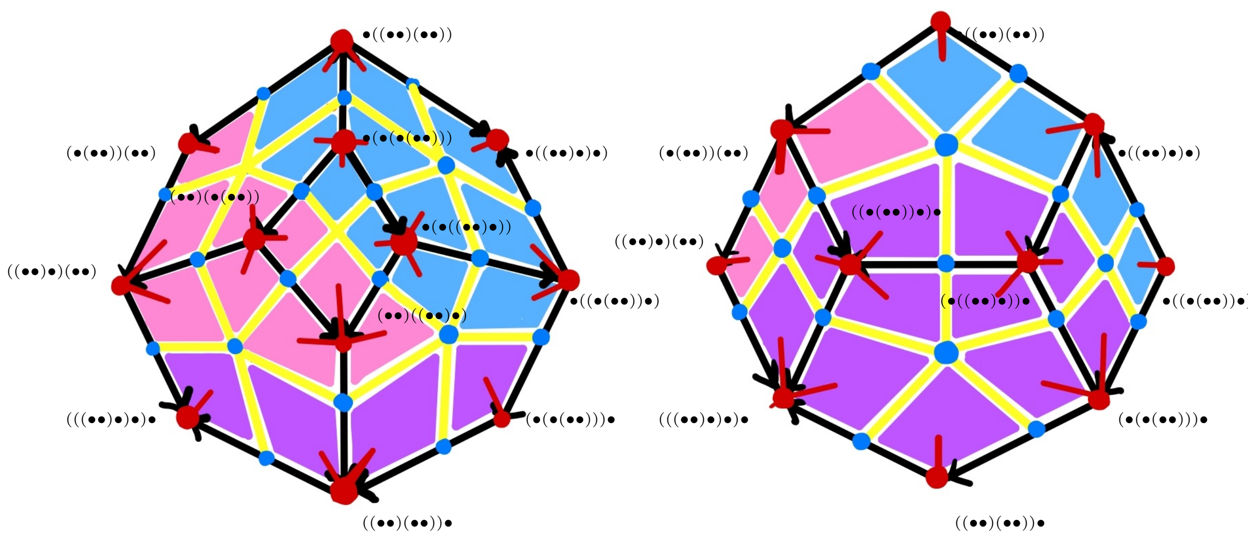


Figure 4.4: Decompostion of the Associahedron in cubes given by images of bracketing trees.

4.5 $\mathbb{E}_2$-algebras on $\mathbb{G}\mathrm{ray}^{\times}_{(3,1)}$: Braided Monoidal 2-categories

In section 4.4, we proved that an $\mathbb{E}_2$-algebra on $\mathbb{C}\mathrm{at}^{\times}_{(2,1)}$ describes and is described by braided monoidal category. Using similar arguments we now study the description of $\mathbb{E}_2$-algebras on $\mathbb{G}\mathrm{ray}^{\times}_{(3,1)}$. Again, we have not defined braided monoidal 2-categories and we choose not to it here. Thus again we opted for an *on the march* approach, where we present our description of a $\mathbb{E}_2$-algebra and at the end arrive to a description which is similar to the definition of braided monoidal 2-categories in the literature. We will omit many arguments and discussion in this section since these are similar to the ones in the proof of theorem 0.0.5.

Remark 4.5.1. A word of caution is needed: The definition of braided monoidal 2-categories has been presented and modified in different ways in the literature. They were originally introduced in [KV94a], and later were modified in [BN96] under the name of semistrict braided monoidal bicategories. In [Gur11] a fully weak version of for braided monoidal bicategories is presented, together with a coherence result. For a nice survey and comparison between the different models we refer the reader to [SP09, §2.1, Appendix C]. The description of $\mathbb{E}_2$-algebras on $\mathbb{G}\mathrm{ray}$ we will present is close to [SD97, Definition 12], where they are referred as braided Gray monoids. This last definition, agrees with the notion of semistrict braided monoidal 2-category as presented in [BN96]. We hope that replacing the tricategory $\mathbb{G}\mathrm{ray}_{(3,1)}$ by $\mathbb{B}\mathrm{icat}_{(3,1)}$ and letting all of the structure to be weak, one may be able get a fully weak definition of braided monoidal bicategories in the style of [SD97] that agrees with the fully weak definition presented in [Gur11].

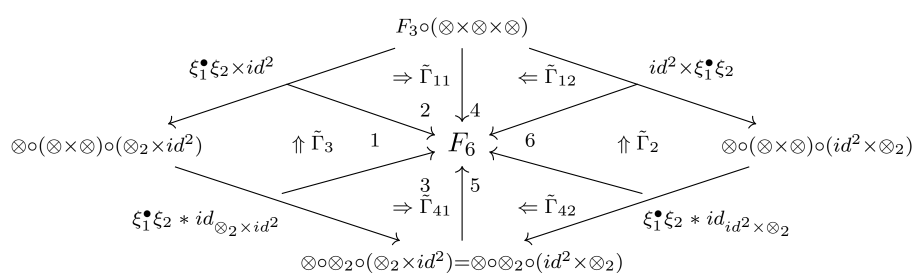
Proof of Theorem 0.0.6. Again, by Dunn's additivity, is enough to consider the images in $\mathbb{G}\mathrm{ray}_{(3,1)}$ of the simplices in $\mathbb{A}\mathrm{ss} \times \mathbb{A}\mathrm{ss}$ described in remark 4.1.11. As in the proof of Theorem 0.0.3 we will assume the cells arising from 0-ary operations are the identity so we will not require to consider unitors. Let $\mathcal{B}$ be the image of the unique object in $(\mathbb{A}\mathrm{ss} \times \mathbb{A}\mathrm{ss})_{\langle 1 \rangle}$.

1. 1-simplices and 2-simplices: The image of trees of length 1 and 2 describe the similar data as in the proof of Theorem 0.0.4: A pair of 2-functors $\otimes : \mathcal{B}\times\mathcal{B} \to \mathcal{B}$ and $I : * \to \mathcal{B}$, together with an adjoint equivalence $\varphi := \xi_1^{\bullet}\xi_2 : \otimes \circ (\otimes_2) \to \otimes \circ (\otimes \times \otimes)$ given on components by

$$\varphi_{(X_1,X_2|X_3,X_4)} : X_1 \otimes X_2 \otimes X_3 \otimes X_4 \to X_1 \otimes X_3 \otimes X_2 \otimes X_4.$$

   As in the proof of Theorem 0.0.4 we will assume the unitors are strict, thus $\varphi_{(X,Y|I,I)} = I = \varphi_{(I,I|Z,W)}$.

2. 3-simplices: Similar as in the proof of Theorem 0.0.4, the images of the trees of length 3 can be arrange into two diagrams:



(4.11)

$$
\begin{tikzcd}[column sep=small]
 & & F_3\circ(\otimes_3) \arrow[dll, "\xi_1^\bullet\xi_2\times id^2"'] \arrow[drr, "id^2\times\xi_1^\bullet\xi_2"] \arrow[d, "4"', "3"] & & \\
\otimes\circ\otimes_2\circ(\otimes\times id)^2 \arrow[rr, "6"] \arrow[drr, "\xi_1^\bullet\xi_2 * id_{\otimes_2\times id^2}"'] \arrow[r, phantom, "\Uparrow\tilde{\Delta}_3"] & & F_6 \arrow[r, phantom, "\Uparrow\tilde{\Delta}_2"] & & \otimes\circ\otimes_2\circ(id\times\otimes)^2 \arrow[ll, "1"'] \arrow[dll, "\xi_1^\bullet\xi_2 * id_{id^2\times\otimes_2}"] \\
 & & \otimes\circ(\otimes\times\otimes)\circ(id\times\otimes)=\otimes\circ(\otimes\times\otimes)\circ(\otimes\times id) \arrow[u, "5", "2"'] & &
\end{tikzcd}
\tag{4.12}
$$

$\Rightarrow \tilde{\Delta}_{11}$, $\Leftarrow \tilde{\Delta}_{12}$, $\Rightarrow \tilde{\Delta}_{41}$, $\Leftarrow \tilde{\Delta}_{42}$

By using the adjuntion data on the pseudonatural transformations appearing in the diagrams 4.11 and 4.12, we can construct modifications as follows:

$$
\begin{tikzcd}[column sep=small]
 & X_1X_2X_3X_4X_5X_6 \arrow[dl, "\varphi_{(X_1,X_2|X_3,X_4)}\otimes id_{X_5\otimes X_6}"'] \arrow[dr, "id_{X_1\otimes X_2}\otimes\varphi_{(X_3,X_4|X_5,X_6)}"] & \\
X_1X_3X_2X_4X_5X_6 \arrow[dr, "\varphi_{(X_1,X_2|X_3\otimes X_5,X_4\otimes X_6)}"'] & \Rightarrow\omega_{(X_1,X_2|X_3,X_4|X_5,X_6)} & X_1X_2X_3X_5X_4X_6 \arrow[dl, "\varphi_{(X_1\otimes X_3,X_2\otimes X_4|X_5,X_6)}"] \\
 & X_1X_3X_5X_2X_4X_6 &
\end{tikzcd}
\tag{4.13}
$$

$$
\begin{tikzcd}[column sep=small]
 & X_1X_4X_2X_5X_3X_6 & \\
X_1X_2X_4X_5X_3X_6 \arrow[ur, "\varphi_{(X_1,X_2|X_4,X_5)}\otimes id_{X_3\otimes X_6}"] & \Leftarrow\theta_{(X_1,X_2,X_3|X_4,X_5,X_6)} & X_1X_4X_2X_3X_5X_6 \arrow[ul, "id_{X_1\otimes X_4}\otimes\varphi_{(X_2,X_5|X_3,X_6)}"'] \\
 & X_1X_2X_3X_4X_5X_6 \arrow[ul, "\varphi_{(X_1\otimes X_2,X_3|X_4\otimes X_5,X_6)}"] \arrow[ur, "\varphi_{(X_1,X_2\otimes X_3|X_4,X_6\otimes X_6)}"'] &
\end{tikzcd}
\tag{4.14}
$$

The strictness of the unit implies that

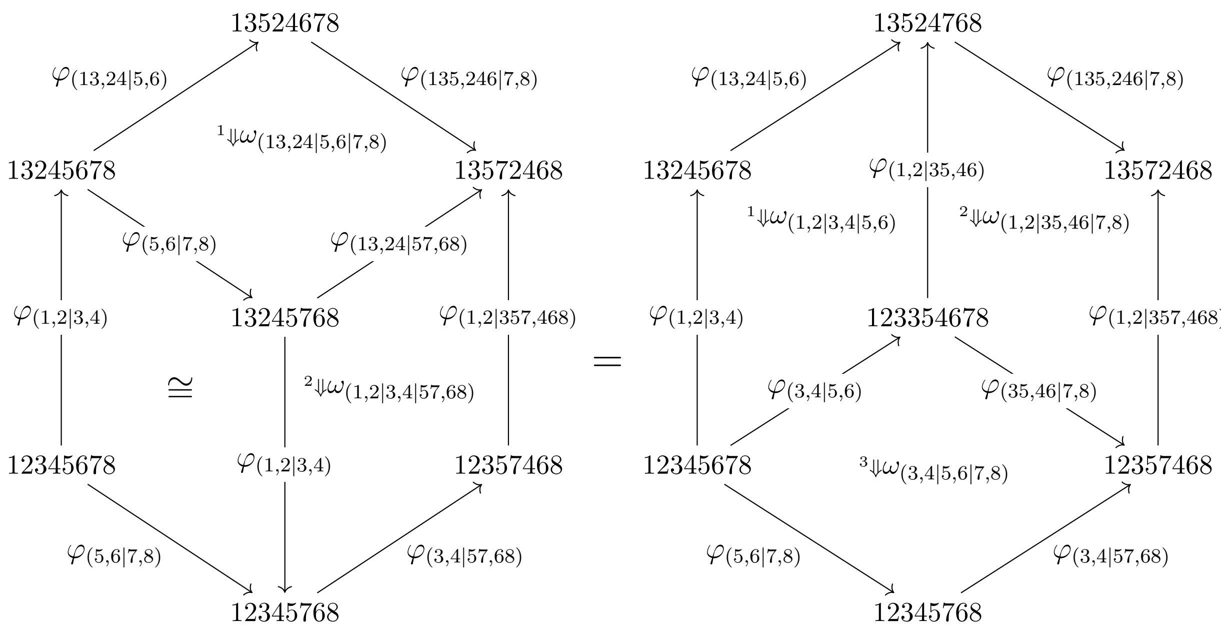
$$I = \omega_{(I,I|X,Y|Z,W)} = \omega_{(X',Y'|I,I|Z',W')} = \omega_{(X'',Y''|Z'',W''|I,I)}$$

and

$$I = \theta_{(I,X,Y|I,Z,W)} = \theta_{(X',Y',I|I,Z',W')} = \theta_{(X'',Y'',Z''|W'',I,I)}.$$

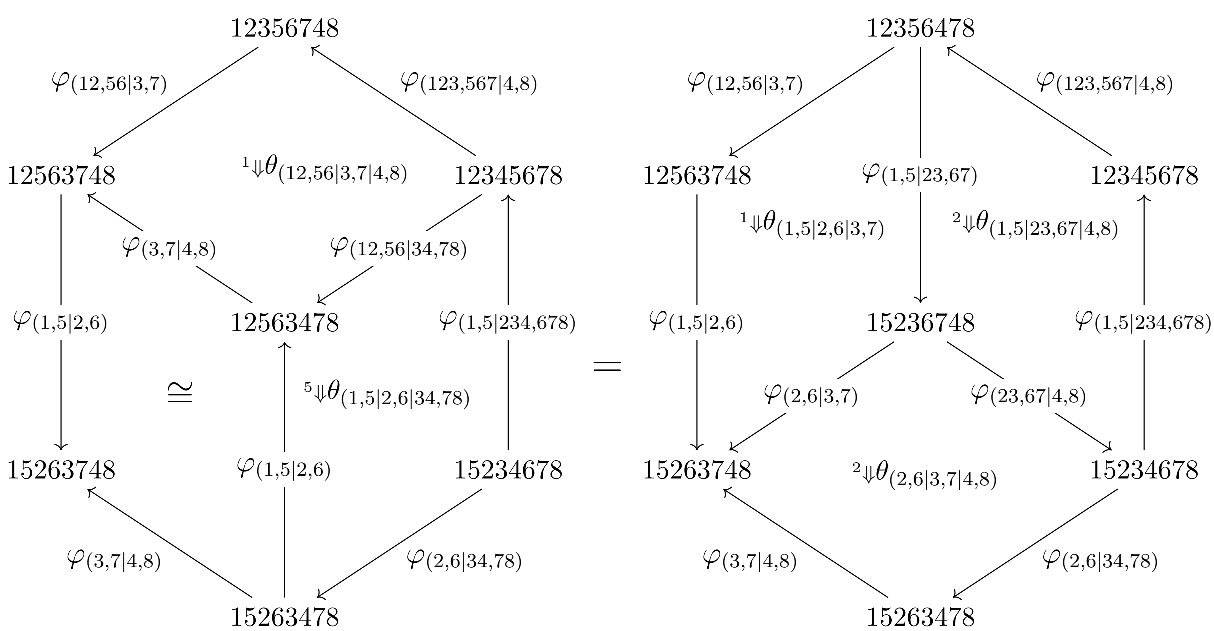
3. 4-simplices: The description of the images of trees of length 4 is given in Appendix A. The procedure to obtain the 4-simplex in $\mathbb{G}$ray from a length 4 tree can be seen in the proof of Theorem 0.0.5. As in the same proof, the coherence data described by all the 4-simplices will be efficiently described by a polytope, or a pasting diagram of 2-cells in the bicategory $\mathbb{G}\mathrm{ray}(\mathcal{B}^m, \mathcal{B})$ (for an appropriate m). Here we will present just the polytopes and pasting diagrams described by the 4-simplices. The verification that all the 4-simplices can be arranged in to the polytopes presented here is carried in Appendix B. All the data can be arranged into 3 polytopes:

   (a) The first polytope appearing is a cube that represents a coherence condition on the modification ω of diagram 4.13



This conditions exhibits the triple $(\otimes, \varphi, \omega)$ as $\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ as a monoidal 2-functor.

(b) The second polytope is an analogous to θ of the condition (a) on ω.



(c) The last polytope has 24 vertices and determines a coherence condition between ω and θ, explicitly can be seen in Figure 4.5. If we cut the polytope through the purple line shown in Figure 4.5, we can also represent the polytope by the pasting diagram

$$
\begin{tikzcd}[column sep=small]
& 142356784 \arrow[rr, "\varphi_{(2,3|5,6)}"] \arrow[drr, "\varphi_{(14,2356|7,89)}"'] \arrow[dr, phantom, "\Downarrow\omega_{(1,23|4,56|7,89)}"] & & 142536789 \arrow[rr, "\varphi_{(14,2356|7,89)}"] \arrow[d, phantom, "\cong"] & & 147253689 \arrow[ddr, "\varphi_{(25,36|8,4)}"] \arrow[dl, phantom, "\Downarrow\omega_{(1,23|4,56|7,89)}"] & \\
& & & 147235689 \arrow[urr, "\varphi_{(2,3|5,6)}"'] \arrow[dr, "\varphi_{(5,6|8,9)}"] \arrow[dd, phantom, "\cong"] & & & \\
12345678 \arrow[uur, "\varphi_{(1,23|4,56)}"] \arrow[rr, "\varphi_{(2,56|7,89)}"] \arrow[dr, "\varphi_{(45,6|78,9)}"'] & & 123475684 \arrow[ur, "\varphi_{(1,23|47,5689)}"] \arrow[dr, "\varphi_{(5,6|8,9)}"] \arrow[d, phantom, "\Downarrow\theta_{(4,5,6|7,8,9)}"] & & 147235869 \arrow[rr, "\varphi_{(2,3|58,69)}"] \arrow[dr, phantom, "\Downarrow\theta_{(1,2,3|47,68,69)}"] & & 147258369 \\
& 123457869 \arrow[rr, "\varphi_{(4,5|7,8)}"'] & & 123475869 \arrow[ur, "\varphi_{(1,23|47,5869)}"] \arrow[rr, "\varphi_{(12,3|4578,69)}"'] & & 124758369 \arrow[ur, "\varphi_{(1,2|47,58)}"'] &
\end{tikzcd}
$$

$$\Vert$$

$$
\begin{tikzcd}[column sep=small]
& 142356784 \arrow[rr, "\varphi_{(2,3|5,6)}"] \arrow[dr, phantom, "\Downarrow\theta_{(1,2,3|47,58,69)}"] & & 142536789 \arrow[rr, "\varphi_{(14,2356|7,89)}"] \arrow[dr, "\varphi_{(1245,36|78,9)}"] \arrow[dd, phantom, "\Downarrow\cong"] & & 147253689 \arrow[dr, "\varphi_{(25,36|8,4)}"] \arrow[dl, phantom, "\Downarrow\theta_{(14,25|36,7|58,69)}"] & \\
12345678 \arrow[ur, "\varphi_{(1,23|4,56)}"] \arrow[rr, "\varphi_{(12,3|45,6)}"] \arrow[ddr, "\varphi_{(45,6|78,9)}"'] & & 123475684 \arrow[ur, "\varphi_{(1,2|4,5)}"] \arrow[dr, "\varphi_{(1245,36|78,9)}"] \arrow[dl, phantom, "\Downarrow\omega_{(12,3|45,6|78,9)}"] & & 147235869 \arrow[rr, "\varphi_{(14,25|7,8)}"] \arrow[dr, phantom, "\Downarrow\omega_{(1,2|4,5|7,8)}"] & & 147258369 \\
& & & 124578369 \arrow[ur, "\varphi_{(1,2|4,5)}"] \arrow[drr, "\varphi_{(4,5|7,8)}"] \arrow[d, phantom, "\cong"] & & & \\
& 123457869 \arrow[urr, "\varphi_{(12,3|4578,69)}"] \arrow[rr, "\varphi_{(4,5|7,8)}"'] & & 123475869 \arrow[rr, "\varphi_{(12,3|4578,69)}"'] & & 124758369 \arrow[uur, "\varphi_{(1,2|47,58)}"'] &
\end{tikzcd}
$$

This ends the description of the data determined by an $\mathbb{E}_2$-algebra on $\mathbb{G}\mathrm{ray}$.

We will now discuss our description with the definitions in [SD97, Definition 12]:

1. Our data of ω and the coherence determined by (a) is exactly those of ω as in [SD97, Definition 12], the only difference is that our maps have 6 indices instead of 4. However this is not relevant since all of our maps act as the identity on the first and last object.

2. In [SD97] the modification θ is assumed to have fixed form in terms of ω. If such form is assumed, then condition (b) in our definition is a consequence of (a).

3. Our Polytope in (c) is related to the second axiom in [SD97, Definition 12]. Indeed we can recover this axiom by setting $X_3 = X$, $X_5 = Y$, $X_7 = Z$ and every other element to be the identity. At the moment the author has not be able to show the equivalence of both definitions. It worth mentioning that this additional axiom was not considered in the first definition of monoidal bicategory in [KV94a] and was actually a surprise to realize that it was necessary for applications of braided monoidal 2-categories (see [Bre94]).

□

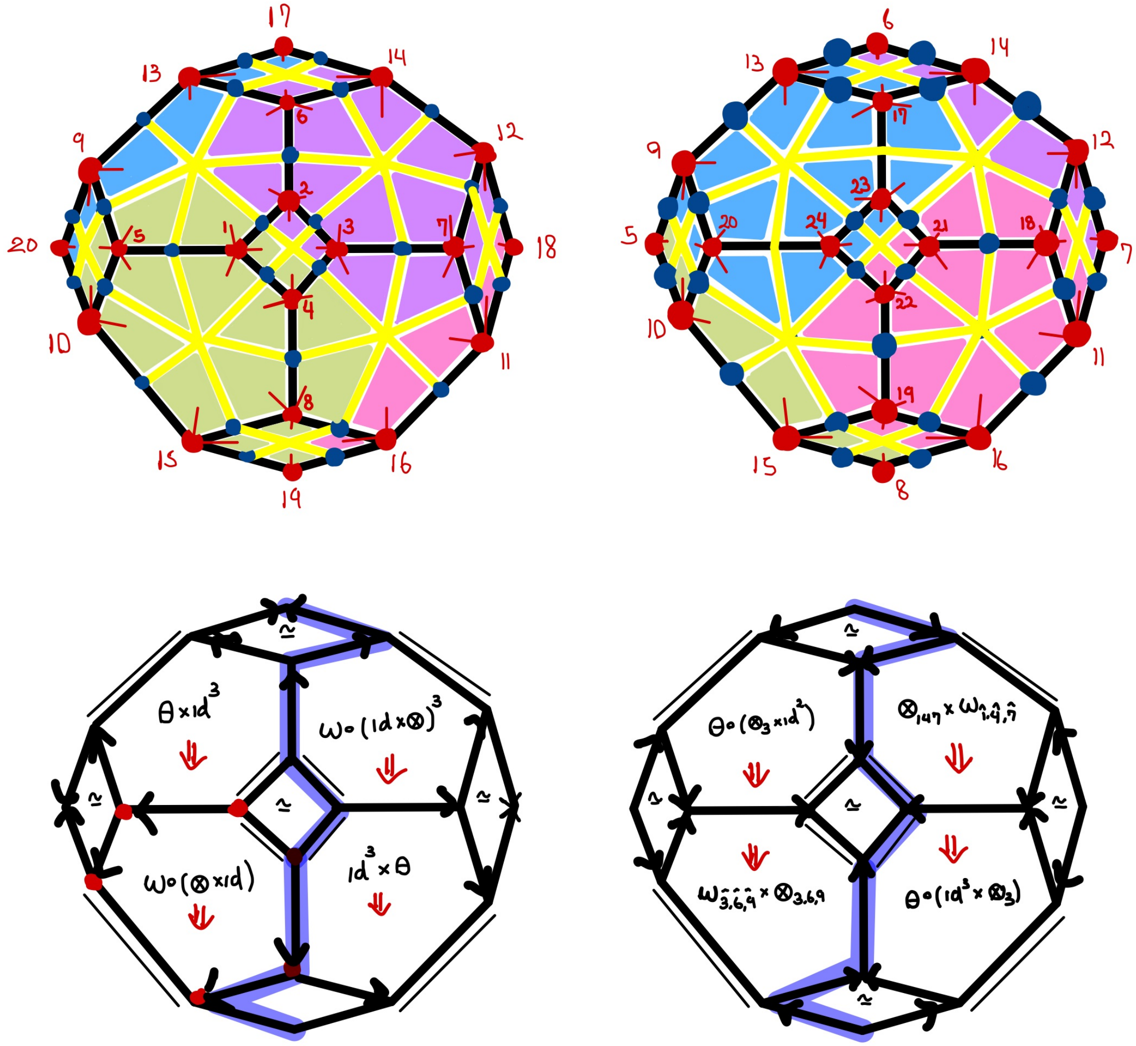


Figure 4.5: On the top: Decomposition into cubes of the polytope describing the coherence condition between ω and θ. On the bottom: labels of the modifications on the faces of the polytope. The purple line defines the exterior of the parting diagram representing this polytope.

4.6 $\mathbb{E}_3$-algebras on $\mathbb{G}\mathrm{ray}^{\times}_{(3,1)}$: Sylleptic Monoidal 2-categories

By the stabilization hypothesis the last non-symmetric $\mathbb{E}_n$-algebra structure in $\mathbb{G}\mathrm{ray}^{\times}_{(3,1)}$) we can consider are $\mathbb{E}_3$ algebras. In this section we will give the definition of a **sylleptic monoidal 2-category** that is mentioned in the statement of conjecture 0.0.7.

Definition 4.6.1. A **sylleptic monoidal 2-category** is given by:

1. A braided monoidal bicategory $(\mathcal{B}, \otimes, \sigma)$.

2. And a invertible modification $\nu : \sigma \Rrightarrow \sigma^{\bullet}$, called a **syllepsis**, and given in components by

$$\begin{tikzcd} X \otimes Y \arrow[rr, bend left, "\sigma_{X,Y}"] \arrow[rr, bend right, "\sigma^{\bullet}_{Y,X}"'] \arrow[rr, phantom, "\Downarrow\nu_{X,Y}"] & & Y \otimes X. \end{tikzcd}$$

Such that the syllepsis satisfies

$$\begin{tikzcd}[column sep=small] XYZ \arrow[rr, bend left, "\sigma^{\bullet}_{Y\otimes Z,X}"] \arrow[rr, "\sigma_{X,Y\otimes Z}"'] \arrow[rr, phantom, bend left=20, "\Uparrow\nu_{X,Y\otimes Z}"] \arrow[dr, bend right, "\sigma_{X,Y}\otimes id_Z"'] \arrow[drr, phantom, "\Uparrow R(X|Y,Z)"] & & YZX \\ & YXZ \arrow[ur, bend right, "id_Y\otimes\sigma_{X,Z}"'] & \end{tikzcd}
=
\begin{tikzcd}[column sep=small] XYZ \arrow[rr, bend left, "\sigma^{\bullet}_{Y\otimes Z,X}"] \arrow[dr, bend right=50, "\sigma_{X,Y}\otimes id_Z"'] \arrow[dr, "\sigma^{\bullet}_{Y,X}\otimes id_Z"] \arrow[rr, phantom, bend right=20, "\Uparrow R^{\bullet}(Y|Z,X)"] & & YZX \\ & YXZ \arrow[ur, "id_Y\otimes\sigma^{\bullet}_{Z,X}"] \arrow[ur, bend right=50, "id_Y\otimes\sigma_{X,Z}"'] \arrow[ul, phantom, bend left=20, "\Uparrow\nu_{X,Y}"] \arrow[ur, phantom, bend right=20, "\Uparrow\nu_{X,Z}"] & \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] XYZ \arrow[rr, bend left, "\sigma^{\bullet}_{Z,X\otimes Y}"] \arrow[rr, "\sigma_{X\otimes Y,Z}"'] \arrow[rr, phantom, bend left=20, "\Uparrow\nu_{X\otimes Y,Z}"] \arrow[dr, bend right, "id_X\otimes\sigma_{Y,Z}"'] \arrow[drr, phantom, "\Uparrow R(X,Y|Z)"] & & ZXY \\ & XZY \arrow[ur, bend right, "\sigma_{X,Z}\otimes id_Y"'] & \end{tikzcd}
=
\begin{tikzcd}[column sep=small] XYZ \arrow[rr, bend left, "\sigma^{\bullet}_{Z,X\otimes Y}"] \arrow[dr, bend right=50, "id_X\otimes\sigma_{Y,Z}"'] \arrow[dr, "id_X\otimes\sigma^{\bullet}_{Z,Y}"] \arrow[rr, phantom, bend right=20, "\Uparrow R^{\bullet}(Z,X|Y)"] & & YZX \\ & YXZ \arrow[ur, "\sigma^{\bullet}_{Z,X}\otimes id_Y"] \arrow[ur, bend right=50, "\sigma_{X,Z}\otimes Y"'] \arrow[ul, phantom, bend left=20, "\Uparrow\nu_{X,Z}"] \arrow[ur, phantom, bend right=20, "\Uparrow\nu_{Y,Z}"] & \end{tikzcd}$$

The ideas leading to the proof of 0.0.7 should be generalizations of the arguments developed in the proofs of Theorem 0.0.5 and Theorem 0.0.6. However the tree combinatorics appearing from using Dunn's additivity become hard to handle. Due to this, at the time of writing, the author has not been able to give a complete proof of conjecture 0.0.7.

4.7 n-Fundamental Groupoids Actions: Braids and Surface Braids

It is well known that a braided monoidal category admits an action of the braid group. To end this chapter we will explain this phenomenon and the associated generalization to higher $(n,1)$-categories from the perspective of $\mathbb{E}_k$-algebras. Suppose $\mathcal{C}$ is a $(n,1)$-category with finite products, and let $\mathbb{E}_k \to \mathcal{C}$ be a lax Cartesian map determining a $\mathbb{E}_k$-algebra structure on $\mathcal{C}^{\times}$. Since $\mathcal{C}$ is an $(n,1)$-category Proposition 2.4.10 implies the $\mathbb{E}_k$-algebra structure is equivalent to a lax Cartesian map from the n-homotopy quasicategory

$$h_n\mathbb{E}_k \to \mathcal{C}.$$

Moreover, by Proposition 2.4.11 this map induces a map between the Kan complexes

$$\tau_n(\mathbb{E}_n(\langle m\rangle, \langle m'\rangle)) \to \mathcal{C}(C^{\times m}, C^{\times m'}),$$

where $C \in \mathcal{C}_0$ is the image of the unique, up to homotopy, object in $\mathbb{E}_n$. Restricting to $m' = 1$, and to the component above $\zeta_m : \langle n\rangle \to \langle 1\rangle$ in the decomposition of $\mathbb{E}_n(\langle m\rangle, \langle 1\rangle)$ we have a map

$$\tau_n(\mathrm{Conf}_m(\mathbb{R}^k)) \to \mathcal{C}(C^{\times m}, C).$$

This map can be considered as an *action* from the homotopy $(n-1)$-groupoid of $\mathrm{Conf}_m(\mathbb{R}^k)$ to the $(n-1)$-category of morphism between the monoidal products $\mathcal{C}^{\times m} \to \mathcal{C}$ determined by the $\mathbb{E}_n$-algebra structure.

Example 4.7.1. Consider the case $\mathcal{C} = \mathbb{C}\mathrm{at}_{(2,1)}$ and $k = 2$. We have a map

$$\tau_2(\mathrm{Conf}_m(\mathbb{R}^2)) \to \mathbb{C}\mathrm{at}_{(2,1)}(C^{\times m}, C).$$

Lets explain in this case how the above map can be seen as an action of the braid group. Since $\tau_2(\mathrm{Conf}_m(\mathbb{R}^2))$ has a unique nontrivial homotopy group in degree 1, then it is homotopy equivalent to $N(\pi_1(\mathrm{Conf}_m(\mathbb{R}^2)))$: the Eilenberg-Maclane space of the group $\pi_1(\mathrm{Conf}_m(\mathbb{R}^2))$. Moreover, the fundamental group $\pi_1(\mathrm{Conf}_m(\mathbb{R}^2))$ is by definition the (unordered) braid group B_m, thus the above map is homotopy equivalent to a map

$$N(B_m) \to \mathbb{C}\mathrm{at}_{(2,1)}(C^{\times m}, C).$$

Using the fact that the nerve of categories is fully faithfull (Theorem 2.4.8), then the above map is equivalent to a functor

$$B_m \to \mathrm{Func}(C^{\times m}, C).$$

And this is equivalent to an action of B_m on the set of natural transformations of $\otimes : C^{\times m} \to C$.

Example 4.7.2. Consider the case $\mathcal{C} = \mathbb{G}\mathrm{ray}_{(3,1)}$ and $k = 3$. We have a map

$$\tau_3(\mathrm{Conf}_m(\mathbb{R}^2)) \to \mathbb{G}\mathrm{ray}_{(3,1)}.$$

From Theorem 2.4.5 it follows that $\tau_3(\mathrm{Conf}_m(\mathbb{R}^2))$ is equivalent to the nerve of a bicategory, in fact it is the nerve of a 2-groupoid which we will denote $\Pi_2(\mathrm{Conf}_m(\mathbb{R}^2))$. The intuitive description of $\Pi_2(\mathrm{Conf}_m(\mathbb{R}^2))$ is as follows: 0-cells are configurations of m points in $\mathbb{R}^2$, 1-cells are braids embedded in $\mathbb{R}^3$ between m points (which should be thought of as homotopies between 0-cells), and 2-cells are surfaces in $\mathbb{R}^4$ with boundary given by braids in $\mathbb{R}^3$ (again these should be thought of as homotopies between 1-cells). In fact in [Gur11] building in [CS98], gave a complete description of generators and relations of the 2-cells in $\Pi_2(\mathrm{Conf}_m(\mathbb{R}^2))$ by means of braid movies and movie moves. Now, using the fact that the nerve of bicategories is fully faithfull (Theorem 2.4.8) the above map is equivalent to a map of bicategories

$$\Pi_2(\mathrm{Conf}_m(\mathbb{R}^2)) \to 2\text{-Funct}(C^{\times m}, C),$$

which by means of [Gur11] can be described by generators are relations.

Remark 4.7.3. Let $\mathcal{C}$ be a $(n,1)$-quasicategory with finite products. The author hopes that in order to define a $\mathbb{E}_k$-algebra on $\mathcal{C}^\times$ it should be enough to define an object $C \in \mathcal{C}_0$ together with a collection of maps $\Pi_n(\mathrm{Conf}_m(\mathbb{R}^k)) \to \mathcal{C}(C^m, C)$ such that they preserve the operad structure. This hope is supported in the fact in the 1-categorical statement, that a map of operads is determined by the map of symmetric sequences between the operation spaces. However, the author is not aware of a complete developed theory of algebras over quaioperads using symmetric sequences, although there is some research on this direction (for example [Hau22]). Assuming a collection of maps $\Pi_n(\mathrm{Conf}_m(\mathbb{R}^k)) \to \mathcal{C}(C^m, C)$ determines an $\mathbb{E}_k$-algebr and assuming there is a finite CW complex description of $\Pi_n(\mathrm{Conf}_m(\mathbb{R}^k))$ at hand, then it would be possible to construct $\mathbb{E}_k$-algebras via a family of *generators* and *relation* on $\mathcal{C}(C^m, C)$ satisfying some compatibility between different m.

Chapter 5

Outlook: Categorification in Representation Theory

We end this thesis with a brief survey on the categorification of fundamental representations of $U_q(\mathfrak{sl}_2)$, and their relationship with $\mathbb{E}_n$-algebras on 2-categories. Moreover, we mention future directions for this work and further connections to topology and TQFTs.

5.1 The Category $\mathrm{Fund}(U_q(\mathfrak{sl}_2))$

To begin this chapter, we will briefly introduce the quantum group $U_q(\mathfrak{sl}_2)$ and its fundamental representation. We follow the conventions of [Jan96], where the reader can find a deeper exposition on the subject. We will always consider the quantum group over the complex numbers $\mathbb{C}$.

Definition 5.1.1. Let $q \in \mathbb{C}$ with $q^2 \neq 1$. The **quantized universal enveloping algebra** $U_q(\mathfrak{sl}_2)$, which we will denote for short U_q in this work, is the (unital associative) algebra with generators E, F, K, K^{-1} and relations

$$\begin{aligned} KK^{-1} &= 1 = K^{-1}K, \\ KE &= q^2 EK, \\ KF &= q^{-2} FK, \\ [E, F] &= \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned}$$

The algebra $U_q(\mathfrak{sl}_2)$ has the structure of a Hopf algebra (see [Jan96, Chapter 3]) with comultiplication $\Delta : U_q \to U_q \otimes U_q$, counit $\epsilon : U_q \to k$, and antipode $S : U_q \to U_q$ given by

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) \ = F \otimes K^{-1} + 1 \otimes F, \quad \Delta(K) = K \otimes K;$$

$$\epsilon(E) = 0, \quad \epsilon(F) = 0, \quad \epsilon(K) = 1;$$
$$S(E) = -K^{-1}E, \quad S(F) \ = -FK, \quad S(K) = K^{-1}.$$

If q is an indeterminate, then U_q behave much like $U(\mathfrak{sl}_2)$, and its representation theory is analogous to the representation theory of $U(\mathfrak{sl}_2)$. For example we can talk about **weight spaces**: common λ-eigenspaces $M_\lambda \subset M$ for the $K^{\pm}$'s, where the eigenvalue λ is called the **weight** of M. Moreover, finite dimensional simple modules are classified by their *highest weight* which are of the form ϵq^a for fixed $\epsilon \in \{\pm 1\}$ and $a \in \mathbb{Z}_{\geq 0}$. We call a module with $\epsilon = 1$ of **type (+1)**, and with $\epsilon = -1$ of **type (-1)**.

Example 5.1.2. The **natural representation** V **of type (-1)** is the two-dimensional U_q-module (over $\mathbb{C}(q)$) with basis v_1, v_0 and action

$$\begin{aligned} Kv_1 &= -q^{-1}v_1, & Ev_1 &= v_0, & Fv_1 &= 0,\\ Kv_0 &= -qv_1, & Ev_0 &= 0, & Fv_0 &= v_1. \end{aligned}$$

Pictorially the representation is given by

$$\begin{tikzcd} K: \quad -q^{-1} \circlearrowleft v_1 \arrow[r, bend left, "E"] & v_0 \circlearrowleft -q \arrow[l, dashed, bend left, "-F"] \end{tikzcd}$$

This is a irreducible finite dimensional representation of weight $-q$, and its weight spaces are $V_{-q} = \mathbb{C}\langle v_1 \rangle$ and $V_q = \mathbb{C}\langle v_0 \rangle$.

Since U_q is a Hopf algebra, then its comultiplication Δ induces a monoidal product on $\mathrm{Rep}(U_q)$. Indeed, for U_q-modules M and N their tensor product $M \otimes_{\mathbb{C}} N$ (as $\mathbb{C}$-modules) is endowed with the U_q-module structure defined by: for $X \in U_q$ the action is

$$X(m \otimes n) := \Delta(X) \cdot (m \otimes n),$$

where $\Delta(X) \in U_q \otimes U_q$ acts by components on $M \otimes N$.

Example 5.1.3. Consider the tensor product of the fundamental representation $V \otimes V$. The action of F on $v_0 \otimes v_0$ is given by

$$F(v_0 \otimes v_0) = F(v_0) \otimes K^{-1}(v_0) + 1(v_0) \otimes F(v_0) = -v_1 \otimes q^{-1}v_0 + v_0 \otimes v_1.$$

In a similar way, we can compute the action of F, E and K on every basis vector. The whole representation can be visualized in the following picture

$$\begin{tikzcd}[column sep=large, row sep=large]
 & v_1 \otimes v_0 \arrow[dl, dashed, shift right, "-q"'] \arrow[dr, shift left, "-q"'] & \\
v_1 \otimes v_1 \arrow[ur, shift right, "q^0"'] \arrow[dr, shift left, "-q^{-1}"] & & v_0 \otimes v_0 \arrow[ul, dashed, shift left, "q^0"'] \arrow[dl, dashed, shift left, "-q^{-1}"] \\
 & v_0 \otimes v_1 \arrow[ul, dashed, shift left, "q^0"] \arrow[ur, shift left, "q^0"] & \\
q^{-2} & q^0 & q^2
\end{tikzcd}$$

In the above picture the action of E is be denoted by a solid arrow while the action of F by dashed arrow. The coefficient over the arrow indicate the target should be multiplied by the given coefficient. Moreover the weight spaces are spanned vectors on the verticals and the corresponding weight can be read at the bottom. In general it can be proven that $V^{\otimes n}$ has weights $-q^2, -q^{2n-2}, \ldots - q^{-2n}$ and that $\dim(V^{\otimes n}|_{q^{2n-2k}}) = \binom{n}{k}$.

Definition 5.1.4. The **category of fundamental representations** $\mathrm{Fund}(U_q(\mathfrak{sl}_2))$, or just $\mathrm{Fund}(U_q)$, is the full subcategory of $\mathrm{Req}(U_q)$ spanned by the representations $V^{\otimes n}$ for $d \geq 0$.

Remark 5.1.5. We can also define the non-quantized category of fundamental representations $\mathrm{Fund}(U(\mathfrak{sl}_2))$ in a completely analogous way. The examples and the discussions presented in this section are almost the same as in the non-quantized upon replacing $-q$ by 1.

The braided monoidal structure in $\mathrm{Fund}(U_q(\mathfrak{sl}_2))$

It is clear that $\mathrm{Fund}(U_q)$ is a full monoidal subcategory of U_q-mod. In this section We will discuss the braiding on U_q-mod and restrict our attention to $\mathrm{Fund}(U_q)$. Let W be a vector space, denote by $\tau : W \otimes W \to W \otimes W$ the flip map $v \otimes w \mapsto w \otimes v$. Let $R \in W \otimes W$, define the elements $R_{12} = R \otimes 1$, $R_{13} = (id \otimes \tau)(R \otimes 1)$, and $R_{23} = 1 \otimes R$ in $W^{\otimes 3}$.

Definition 5.1.6. Let H be a Hopf algebra and $R \in H \otimes H$. The pair (H, R) is called a **quasitriangular Hopf algebra** if R is invertible and if for any $a \in H$ it holds that

$$\begin{aligned} \tau(\Delta(a))R &= R\Delta(a), \\ (\Delta \otimes id_H)(R) &= R_{13}R_{23}, \\ (id_H \otimes \Delta)(R) &= R_{13}R_{12}. \end{aligned} \tag{5.1}$$

We call R the **universal R matrix** of H.

Considering elements in $H^{\otimes n}$ as maps on itself via left multiplication, then the conditions in 5.1 are given by the diagrams

$$\begin{tikzcd} H \otimes H \arrow[r, "\Delta(a)"] \arrow[d, "R"] & H \otimes H \arrow[d, "R"'] \\ H \otimes H \arrow[r, "\Delta^{op}(a) := \tau(\Delta(a))"'] & H \otimes H \end{tikzcd}$$

$$\begin{tikzcd} H \otimes H \otimes H \arrow[r, "(\Delta \otimes id_H)R"] \arrow[d, "R_{23} = 1 \otimes R"'] & H \otimes H \otimes H \\ H \otimes H \otimes H \arrow[ur, "R_{13} = (id \otimes \tau)(R \otimes 1)"'] & \end{tikzcd} \qquad \begin{tikzcd} H \otimes H \otimes H \arrow[r, "(id_H \otimes \Delta)R"] \arrow[d, "R_{12} = R \otimes 1"'] & H \otimes H \otimes H \\ H \otimes H \otimes H \arrow[ur, "R_{13} = (id \otimes \tau)(R \otimes 1)"'] & \end{tikzcd}$$

The diagrams in 5.1 resemble the axioms of a braided monoidal category (Definition 4.2.1): the first diagram is associated to naturality, while the diagrams on the bottom are the compatibility of the braiding. In fact one can prove that $\tau \circ R : H \otimes H \to H \otimes H$ is a Hopf algebra morphism and defines a braiding in the category of finite dimensional H-modules. For proofs of all these facts we refer the reader to [Kas12, Chapter VIII]. Our quantum algebra U_q does not have an universal R-matrix *per se*, nevertheless upon considering the completion of U_q there is an universal R-matrix $R \in U_q \hat{\otimes} U_q$ given by the formal sum

$$R = q^{\frac{K \otimes K}{2}} \sum_{n=0}^{\infty} \frac{(q - q^{-1})^n}{[n]_q!} q^{-n(n-1)/2} F^n \otimes E^n, \tag{5.2}$$

where $q^{\frac{K \otimes K}{2}}(v \otimes w) = q^{\frac{\lambda\mu}{2}} v \otimes w$ and $[n]_q!$ is the quantum factorial (see [Kas12]). Notice that on a finite dimensional U_q-module M just finitely many of the summands in (5.2) act as nonzero operators (since F^n act by zero on a finite dimensional U_q-modules for $n >> 0$). Therefore, on finite dimensional modules R is a well defined U_q-module morphism, and the map $\tau \circ R$ defines a braiding in U_q-mod. In particular this braiding restricts to $\text{Fund}(U_q)$ endowing the latter with the structure of a braided monoidal category.

Example 5.1.7. We will give and example of the braiding acting on $V \otimes V$ the tensor product of the fundamental representation with itself. Lets order its basis by $\{v_1 \otimes v_1, v_0 \otimes v_1, v_1 \otimes v_0, v_0 \otimes v_0\}$, and calculate the action of R on a basis vector

$$R(v_0 \otimes v_1) = (1 \otimes 1 + (q - q^{-1})F \otimes E)(v_0 \otimes v_1) = v_0 \otimes v_1 + (q - q^{-1})v_1 \otimes v_0.$$

Similarly one can compute the action on the other basis vectors and obtain that the action of R in $V \otimes V$ is given by the matrix

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & q & 0 \end{pmatrix}.$$

Composing with the flip map we obtain that the braiding action on $V \otimes V$ is given by

$$\tau \circ R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & q & 0 \end{pmatrix}.$$

From the discussion developed on Chapter 4 on the description of $\mathbb{E}_n$-algebras on $\mathbb{C}\text{at}$ we can conclude the following:

- **The category of fundamental representations of $U_q(\mathfrak{sl}_2)$ is $\mathbb{E}_2$-algebra on $\mathbb{C}\text{at}^\times$.**

5.2 Categorification of the Fundamental Representations

Let A be a monoidal abelian category, then the **Grothendick group** of A is

$$K_{\mathbb{C}}(A) = K(A) \otimes_{\mathbb{Z}} \mathbb{C},$$

where $K(A)$ is the free $\mathbb{Z}$-module generated by the symbols $[M]$, where M is an object of A, subject to the relations $[M] = [M'] + [M'']$ for all short exact sequences

$$0 \to M' \to M \to M'' \to 0.$$

Notice that by definition an exact functor $F : A \to A'$ of abelian categories defines a linear map $K(F) : K_{\mathbb{C}}(A) \to K_{\mathbb{C}}(A')$. Now we can give a first idea of what is categorification. By a **categorification** of a U_q-module M we mean:

1. An abelian category A,
2. exact endofunctors $\tilde{E}, \tilde{F}, \tilde{K}^{\pm}$,
3. an isomorphism $\phi : K_{\mathbb{C}}(A) \cong M$, such that under the isomorphism ϕ the morphisms $K(\tilde{E}), K(\tilde{F}), K(\tilde{K}^{\pm})$ act as $E, F, K^{\pm}$.

In general *categorification* is a bigger program in which one hope to upgrade all the structure of an U_q-module to the categorical level. A bit more precisely, is the construction of categories and 2-categories with given Grothendieck rings, together with exact functors and short exact sequences inducing given morphisms and equations in the desired Grothendieck rings. In the previous section we introduced the category $\mathrm{Fund}(U_q)$ and described its structure as a braided monoidal category. In this section we briefly discuss the categorification for the modules $V^{\otimes}$. Moreover we briefly discuss how these fit in a *naive* 2-categorification of $\mathrm{Fund}(U_q)$. We use the adjective naive because a complete categorification requires to upgrade all the data of $\mathrm{Fund}(U_q)$, like the action of $E, F, K^{\pm}$, their the Hopf algebra structure, the braiding, etc., to the 2-categorical level, which is beyond the presentation of this work.

Categorification of Fund($\mathfrak{sl}_2$) via Category $\mathcal{O}$

In this section we will briefly present the categorification of representations in $\mathrm{Fund}(U(\mathfrak{sl}_2))$ via category $\mathcal{O}$, as developed in [BFK00]. For a general discussion on the structure of category $\mathcal{O}$ we refer the reader to [Hum08]. We will mainly be interested in type A semisimple lie algebras, thus in what follows $\mathfrak{g} = \mathfrak{sl}_n$ for some $n \in \mathbb{N}$.

Preliminaries of category $\mathcal{O}$

We will quickly introduce all the notation and necessary facts for the later discussions. To define category $\mathcal{O}$ fix a Cartan and Borel subalgebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ inducing a Cartan decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. If we choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$, then Cartan and Borel algebras in $\mathfrak{sl}_n$ are given by diagonal matrices and upper triangular matrices respectively. We denote the universal enveloping algebras of the previous lie algebras by $U(\mathfrak{g})$, $U(\mathfrak{h})$, etc. Moreover, we denote the center of $U(\mathfrak{g})$ by $\mathcal{Z}(\mathfrak{g})$.

Definition 5.2.1. The **BGG category** $\mathcal{O}(\mathfrak{g})$, also referred just as $\mathcal{O}$ when $\mathfrak{g}$ is clear from the context, is the full subcategory of $U(\mathfrak{g})$-mod given by finitely generated, locally $U(\mathfrak{n}_+)$-finite, and $\mathfrak{h}$-diagnonalizable modules. We say M is $U(\mathfrak{n}_+)$-finite if $U(\mathfrak{n}_+) \cdot m$ is finite dimensional for every $m \in M$, and M is $\mathfrak{h}$-diagonalizable if it has a weight space decomposition $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$.

We call a $U(\mathfrak{g})$-module M **highest weight** with weight λ if there exist a vector $v \in M$ with weight λ such that M is $U(\mathfrak{n}_-)$-generated by a single element v^+, and $U(\mathfrak{n}_+)v^+ = 0$; the vector v^+ is called the **maximal weight vector**. It is easy to see that highest weight modules belong to $\mathcal{O}$ and that their λ-weight space M_λ is 1-dimensional. In particular **Verma modules** $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ are in $\mathcal{O}$. Now $\mathcal{O}$ is closed under submodules, quotients, and finite direct sums (see [Hum08, §1.3]). In consequence the unique simple quotient $L(\lambda)$ of $M(\lambda)$ also belong to $\mathcal{O}$; in fact every the simple modules in $\mathcal{O}$ is isomorphic to some $L(\lambda)$. Even more, every element in $\mathcal{O}$ has a Jordan-Hölder composition series of finite length with subquotients $L(\lambda)$, which implies $[L(\lambda)]$ form a basis for the Groethendieck group of $\mathcal{O}$.

Taking a small detour from category $\mathcal{O}$. An algebra homomorphism $\chi : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ is called a **central character**. It is easy to see that central characters are in correspondence with the maximal ideals J_χ of $\mathbb{Z}(\mathfrak{g})$. Recall that the Weyl dot action is defined as $w \bullet \lambda = w(\lambda + \rho) - \rho$ for $\omega \in W$ and $\lambda \in \mathfrak{h}^*$. The Harish-Chandra isomorphism states that

$$\mathcal{Z}(\mathfrak{g}) \cong S(\mathfrak{h})^{W_\bullet},$$

where $S(\mathfrak{h})^{W_\bullet}$ are the invariants of the Weyl dot action on the polynomial algebra $S(\mathfrak{h})$ on $\mathfrak{h}$ (see [Hum08, §1.9]). By the Harish-Chandra isomorphism it follows that central characters are labeled by the Weyl-dot orbits in $\mathfrak{h}$. Thus to every $\lambda \in \mathfrak{h}$ there is an associated central character $\chi(\lambda)$ such that $\chi_\lambda = \chi_{w\bullet\lambda}$ for every $w \in W$, and every character is of this form.

Going back to category $\mathcal{O}$. Let $\mathcal{O}_\lambda$ be the full subcategory of $\mathcal{O}$ consisting of modules M such that $J^n_{\chi(\lambda)}M = 0$ for $n >> 0$. We call the categories $\mathcal{O}_\lambda$ the **blocks** of $\mathcal{O}$, in particular we call $\mathcal{O}_0$ the **principal block**. By usual linear algebra, any module M in $\mathcal{O}$ has a generalized eigenspace decomposition with respect to the action of the center, thus $\mathcal{O}$ decomposes as a direct sum of its blocks $\mathcal{O} = \mathcal{O}_\lambda$. It is not hard to check that $\mathcal{O}_\lambda$ is closed under taking quotients and extensions, thus $\mathcal{O}_\lambda$ can also be described as the full subcategory of $\mathcal{O}$ consisting of the modules M with the property that all its simple subquotients are isomorphic to $L(\omega \bullet \lambda)$ for some $w \in W$. By construction, the simples in $\mathcal{O}_\lambda$ are of the form $L(w \bullet \lambda)$ for $w \in W$, thus $\{[L(w \bullet \lambda)]\}_{w\in W}$ forms a basis of $K_\mathbb{C}(\mathcal{O}_\lambda)$. Moreover, one can prove that the classes of the Vermammodules $\{[M(w \bullet \lambda)]\}_{w\in W}$ also form a basis of the Grothendieck group (see [Hum08, §3.11]), called the **standard basis**.

The construction of the Categorification

Recall that for $\mathfrak{sl}_n$ the Weyl group is isomorphic by S_n and it acts on the Cartan $\mathfrak{h} \cong \mathbb{C}^n$ by permutation of the basis vectors $\{e_1, \dots, e_n\}$. Let $\lambda_k \in \mathfrak{h}^*$ be a weight such that its stabilizer is $S_k \times S_{n-k}$ (for example $\lambda_k = e_1 + \cdots + e_k$), then we denote

$$\mathcal{O}_{(n,n-k)} := \mathcal{O}_{\lambda_k}.$$

The Grothendieck group of $\mathcal{O}_{(n,n-k)}$ has dimension $\binom{n}{k}$ (the cardinality of $S_n/(S_k \times S_{n-k})$), which coincides with the dimension of the k-weight space of $V^\otimes$. Now for $\lambda_k = e_1 + \cdots + e_k$ every weight $\omega \cdot \lambda_k$ is of the form $a^\omega_1 e_i + \cdots + a^\omega_n e_k$ for a binary sequence $(a^\omega_1, \dots, a^\omega_n)$ with exactly k ones (see [BFK00]). Therefore there is an isomorphism of vector spaces.

$$K_\mathbb{C}\left(\bigoplus_{k=0}^{n} \mathcal{O}_{n,n-k}\right) \cong \bigoplus_{k=0}^{n} V_k^{\otimes n} = V^{\otimes n}, \quad M(\omega \cdot \lambda_k) \mapsto v_{a^\omega_1} \otimes v_{a^\omega_n},$$

where $\{v_0, v_1\}$ are the basis vectors of the fundamental representation of $U(\mathfrak{sl}_2)$. We will denote

$$\mathcal{O}_n := \bigoplus_{k=0}^{n} \mathcal{O}_{n,n-k},$$

and refer to it as the categorification of the vector space $V^{\otimes n}$.

To be completely precise, for the categorification of the module structure one needs to consider Luztig's version $\dot{U}(\mathfrak{sl}_2)$ of $U(\mathfrak{sl}_2)$, see [BFK00] for details of the definition of $\dot{U}(\mathfrak{sl}_2)$. Here we only mention that the generators $\dot{U}(\mathfrak{sl}_2)$ are divided powers of E and F (which we will denote with over dots), with relations adequately normalized, and instead of a unit it has a system of idempotents 1_n for $n \in \mathbb{Z}$ which behave projectors onto integral weight spaces. Without worrying to much on the details we can describe the categorification of $\dot{E}$, $\dot{F}$ and $\dot{H}$ on $\mathcal{O}_n$. A functor is called **projective** if it is exact and maps projectives objects to projectives objects. The basic example of projective functors are **translation functors** $\theta^\mu_\lambda : \mathcal{O}_\lambda \to \mathcal{O}_\mu$ given by

$$\theta^\mu_\lambda : \mathrm{Pr}_\lambda \circ (L(\mu) \otimes -),$$

where Pr_λ is the projection onto the block $\mathcal{O}_\lambda$ and $L(\mu) \otimes -$ is the functor given by tensoring with the finite dimensional representation $L(\mu)$ for some μ dominant integral. In fact every projective functor is isomorphic to a direct summand of θ_λ^μ for some λ and μ (see [BG80]). For each k define

$$\begin{aligned}
\mathcal{E}_i &:= \theta_{\lambda_k}^{\lambda_{k+1}} : \mathcal{O}_{k,n-k} \to \mathcal{O}_{k+1,n-k-1}, \\
\mathcal{F}_i &:= \theta_{\lambda_k}^{\lambda_{k-1}} : \mathcal{O}_{k,n-k} \to \mathcal{O}_{k-1,n-k+1}.
\end{aligned}$$

Since these functors are exact they induce linear maps on the Grothendieck groups

$$\begin{aligned}
[\mathcal{E}_i] &: K_{\mathbb{C}}(\mathcal{O}_{k,n-k}) \to K_{\mathbb{C}}(\mathcal{O}_{k+1,n-k-1}), \\
[\mathcal{F}_i] &: K_{\mathbb{C}}(\mathcal{O}_{k,n-k}) \to K_{\mathbb{C}}(\mathcal{O}_{k-1,n-k+1}).
\end{aligned}$$

Theorem 5.2.2. $\mathcal{O}_n$ together with $\mathcal{E}$ and $\mathcal{F}$ form a categorification of $V^{\otimes n}$ as a $\dot{U}(\mathfrak{sl}_2)$-module.

Proof. This is main theorem of [BFK00], where the reader may find details of the construction. □

Example 5.2.3. Let $n = 2$ so $\mathfrak{g} = \mathfrak{sl}_2$, then we can either consider $k = 0, 1$ or 2. For $k = 0$ or 2: we have that $-\rho$ is a weight fixed by the Weyl dot action, thus $\mathcal{O}_{k,n-k} \cong \mathcal{O}_{n-k,k}$ is generated by one simple object $L(-\rho) = M(-\rho)$. On the other hand, for $k = 1$: The weight 0 is regular the stabilizer is trivial, and $\mathcal{O}_{1,1}$ has two Verma modules $M(0)$ and $M(s \bullet 0)$. Computing the action of $\theta_0^{-\rho}$ and $\theta_{-\rho}^0$ requires more theory from category $\mathcal{O}$ than the one we can present here. In general to compute translation functors on Verma modules one may use [Hum08, Theorem 7.6 and Theorem 7.12], in the case of $\mathfrak{sl}_2$ this can be summarized in

$$\theta_{-\rho}^0 M(0) = M(-\rho), \quad \theta_0^{-\rho} M(s \bullet 0) = M(-\rho),$$

and the fact that $\theta_0^\rho M(-\rho)$ has a Verma flag with composition factors $M(0)$ and $M(s \bullet 0)$ each appearing once. We can arrange the direct summands of $\mathcal{O}$ and the actions of $\mathcal{E}, \mathcal{F}$ as follows

$$\begin{tikzcd}
& {[M(s \bullet 0)]} \arrow[dl, dashed, bend right, "\theta_0^{-\rho}"'] \arrow[dr, "\theta_0^{-\rho}"] & \\
{[M(-\rho)]} \arrow[ur, "\theta_{-\rho}^0"'] \arrow[dr, "\theta_{-\rho}^0"] & & {[M(-\rho)]} \arrow[ul, dashed, bend right, "\theta_{-\rho}^0"'] \arrow[dl, dashed, bend left, "\theta_{-\rho}^0"] \\
& {[M(0)]} \arrow[ul, dashed, bend left, "\theta_0^{-\rho}"] \arrow[ur, "\theta_0^{-\rho}"'] &
\end{tikzcd}$$

$$\mathcal{O}_{2,0} \qquad\qquad\qquad \mathcal{O}_{1,1} \qquad\qquad\qquad \mathcal{O}_{0,2}$$

Where we wrote the associated stabilizer of S_n describing the category $\mathcal{O}_{n,n-k}$ on the bottom. Moreover, the action of $[\mathcal{E}]$ is given by solid arrows and the action of $[\mathcal{F}]$ by dashed arrows.

We could organize all of the categorifications for $V^{\otimes n}$ into a naive 2-categorification of $\mathrm{Fund}(U(\mathfrak{sl}_2))$, which we will denote $\mathcal{O}_\bullet$: 0-cells are given by natural numbers $n \in \mathbb{N}$ (which should be considered as associated to $\mathcal{O}_n$), 1-cells are given by projective functors $F_n^m : \mathcal{O}_n \to \mathcal{O}_m$, and

2-cells are natural isomorphisms of projective functors. In view of the categorification picture, and since $\mathrm{Fund}(U(\mathfrak{sl}_2))$ is a monoidal category, then we expect the following:

- **The 2-category $\mathcal{O}_\bullet$ can be endowed with the structure of an $\mathbb{E}_1$-algebra on $\mathbb{G}$ray$^\times$.**

To consider the categorification of $\mathrm{Fund}(U_q)$ one need to consider the graded lifts $\mathcal{O}_n$, $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{F}}$ for the categories $\mathcal{O}$ and the functors $\mathcal{E}$ and $\mathcal{F}$. Intuitively the grading plays the role of q in the Grothendieck ring. For a in depth discussion of graded lifts and their role in categorification we refer the reader to [Str05]. Moreover, to categorify the action of the Hecke algebra on $V^{\otimes n}$, which is intimately related to the braiding, it is necessary to consider derived Zuckerman functors (see [Str05]). Thus we might need to consider the 2-category $D_b(\mathcal{O}_\bullet)$: 0-cells are given by natural numbers $n \in \mathbb{N}$ (which should be considered as associated to $\mathcal{O}_n$), 1-cells are given by derived projective functors $DF_n^m : D_b(\mathcal{O}_n) \to D_b(\mathcal{O}_m)$, and 2-cells are natural isomorphisms of derived projective functors. Again in view of the categorification picture, and since $\mathrm{Fund}(U_q)$ is a braided monoidal category, then we expect the following:

- **The 2-category $D_b(\mathcal{O}_\bullet)$ can be endowed with the structure of an $\mathbb{E}_2$-algebra on $\mathbb{G}$ray$^\times$.**

It is also possible that in the quasicategory framework it is not necessary to consider derived categories, but instead work with a truncation of the dg-category of chain complexes in $\mathcal{O}$.

Categorification of Fund(U_q) via Geometric Bimodules

We will now discuss the categorification of $\mathrm{Fund}(\mathfrak{sl}_2)$ based on Khovanov's arc algebras following [BS08] and [Kho02]. Khovanov's arc algebras are finite dimensional graded algebras K_n, indexed by $n \in \mathbb{N}$, build from the 2-dimensional TQFT associated to the Frobenius algebra $\mathbb{C}[x]/(x^2)$. Similarly, geometric bimodules are graded (K_n, K_m)-bimodules associated to a (n, m)-tangle diagram and the aforementioned 2-dimensional TQFT. All geometric bimodules organize in a 2-category denoted Kbim of geometric bimodules: 0-cells are given by natural numbers $n \in \mathbb{N}$ (which should be considered as associated to K_n-grmod), 2-cells are given by finitely generated projective graded (K_n, K_m)-bimodules, and 2-cells are bimodules isomorphisms. This 2-category can be considered an algebraic analogue of **graded** maximal parabolic category $\mathcal{O}$. Let us take a moment to explain this statement, for which we will have to discuss in a little more detail the parabolic category $\mathcal{O}$.

Let $S \subset \Delta$ be a subset of the simple roots. The subset S determines a parabolic subalgebra $\mathfrak{p}_S$ containing $\mathfrak{b}$ with enveloping algebra $U(\mathfrak{p})$. The parabolic Weyl group W_S of $\mathfrak{p}_S$ is generated by s_α for $\alpha \in S$. Parabolic subalgebras of $\mathfrak{sl}_n$ are given by block upper triangular matrices, in particular we denote by $\mathfrak{p}_{k,n-k}$ the parabolic subalgebras of $\mathfrak{sl}_n$ consisting of block upper triangular matrices with 2-block rows of height k and $n-k$. We remark that the parabolic Weyl groups for $\mathfrak{p}_{k,n-k}$ are given by $S_k \times S_{n-k}$. Now, similar to the definition of category $\mathcal{O}$ we can use the Levi decomposition determined by a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ to define a parabolic category $\mathcal{O}^{\mathfrak{p}}$ (see [Hum08, Chapter 9]). This is a the full subcategory of $\mathcal{O}$ given by $U(\mathfrak{p})$-locally finite modules. In the case of 2-block rows parabolics of $\mathfrak{sl}_n$ we will denote by $\mathcal{O}^{(k,n)} := \mathcal{O}^{\mathfrak{p}_{k,n-k}}$. Similar to the case of the categories $\mathcal{O}_{k,n-k}$ the categories $\mathcal{O}^{k,n-k}$ have $\binom{n}{k}$ simple modules. It is possible to use the category $\bigoplus \mathcal{O}^{k,n-k}$ to categorify $\mathrm{Fund}(U(\mathfrak{sl}_2))$, in this case the translation functors are replaced by (derived) Zuckerman functors for the action of $\mathcal{E}$ and $\mathcal{F}$ (see [BFK00]). The two descriptions are related by a Koszul duality between parabolic category $\mathcal{O}$ and the singular blocks of $\mathcal{O}$ (see [BGS96]). Now, with this Koszul duality in mind we can relate $\mathcal{O}_\bullet$ and Kbim via the following theorems:

Theorem 5.2.4 ([BS11])**.** There is an equivalence of categories

$$\mathcal{O}^{(i,n-i)} \xrightarrow{\sim} K^{(i,n-i)}\text{-mod},$$

such that simples are mapped to simples. In particular, this equivalence restrict to the principal blocks and summing over all $(i, n-i)$ there is an equivalence

$$\bigoplus_{i=0}^{n} \mathcal{O}_0^{(i,n-i)} \xrightarrow{\sim} \bigoplus_{i=0}^{n} K_0^{(i,n-i)}\text{-mod}.$$

Theorem 5.2.5 ([Wat60])**.** Let A, B be algebras. To each (A, B)-bimodule R there is an associated functor

$$- \otimes R : B\text{-mod} \to A\text{-mod}, \qquad M \mapsto M \otimes_B R$$

. The following properties hold:

1. If R is a projective and finitely generated (A, B)-bimodule then the functor $- \otimes R$ is projective.

2. Every projective functor $F : A$**-mod** $\to B$**-mod** which preserves limits is equivalent to $- \otimes R$ for a projective finitely generated (A, B)-bimodule R.

3. If R, S are (A, B)-bimodules, then

$$A\text{-mod-}B(R, S) \cong \mathrm{Func}(- \otimes R, - \otimes S),$$

   where the set on the right is the set of natural transformations between the functors $- \otimes R$ and $- \otimes S$.

We remark that Khovanov arc algebras and geometric bimodules come naturally with gradings, implying that the categories $K^{(i,n-i)}$-grmod of graded modules come equipped with a graded lift of the categories of (ungraded) modules $K^{(i,n-i)}$-mod (see [BS08] for details). Again following the categorification intuition we expect the following:

- **The 2-category Kbim can be endowed with the structure of an $\mathbb{E}_1$-algebra on $\mathbb{G}$ray$^{\times}$.**

Moreover, passing to chain complexes we can consider crossings in tangle diagrams, which are represented by 2-term chain complexes of bimodules (see [Kho02]). Thus to consider an $\mathbb{E}_2$-algebra structure is seem necessary to consider the 2-category $\widehat{\mathrm{Kbim}}$: 0-cells are natural numbers $n \in \mathbb{N}$, 1-cells are chain complexes of (K_m, K_n)-geometric bimodules, and 2-cells are isomorphisms of chain complexes.

- **The 2-category $\widehat{\mathbf{Kbim}}$ can be endowed with the structure of an $\mathbb{E}_2$-algebra on $\mathbb{G}$ray$^{\times}$.**

Constructing the $\mathbb{E}_n$-algebras structures on the 2-categories presented in this chapter will be part of a follow up work to this thesis.

At last, we finish this work with a small discussion on how explicit examples of $\mathbb{E}_n$-algebras on weak n-categories can be applied in low dimensional topology.

5.3 Low Dimensional Topology and TQFTs

We end this thesis with and outlook on the connections of this work with TQFTs and mathematical approximations to quantum field theory. In this work we presented just the basic example of factorization algebras, i.e. $\mathbb{E}_n$-algebras. However there are more general factorization algebras that would allow to better understand the interplay between representation theory and low dimensional topology from a higher categorical perspective.

There are variations of the $\mathbb{E}_n$-operad for each subgroup $H \subset \mathrm{Top}(k) := \mathrm{Aut}(\mathbb{R}^k)$. These variations are related to restrictions of tangential structure of $\mathbb{R}^n$. For example, $\mathbb{E}_n$-operads studied in this work correspond to the subgroup $\langle e \rangle \subset \mathrm{Top}(k)$ and to complete framings of $T\mathbb{R}^n$. This was a consequence of considering rectilinear embedding as our operation spaces for the $\mathbb{E}_n$-operad. For the subgroup $\mathbb{Z}_2 \subset O(n) \subset \mathrm{Top}(k)$, given by reflections, the associated operad has spaces of operations given by $SO(n)$-embeddings of disks (rectilinear plus we allow rotations of disks). In the case $n = 2$ the associated quasioperad is equivalent to $\mathbb{E}_2 \rtimes \mathbb{Z}_2$. Similar to the result presented in this work, but with different techniques, in [Wah01] it is shown that ribbon categories are equivalent to $\mathbb{E}_2 \rtimes \mathbb{Z}_2$-algebras in categories.

On the other hand, Reshetikhin-Turaev in [Tur92] used a dualizable and nondegenerate version of ribbon categories, called modular categories, as their main input for constructing invariants of 3-manifolds. The Reshetikhin-Turaev invariants were motivated by the invariants obtained by Witten through quantum Chern-Simmon theory on 3-manifold, and are expected to be the mathematical formalization of them. Nevertheless, there is not a fully explicit proof relating both of these invariants. Costello and Gwilliams in [CG21] (chapter 8) construct a locally constant factorization algebra over $\mathbb{R}^3$, i.e. an $\mathbb{E}_3$-algebra, for the $U(1)$-Chern-Simmon theory on $\mathbb{R}^3$. Moreover they show that the homotopy category of (left) modules, which by Dunn additivity has the structure of an $\mathbb{E}_2$-algebra, is equivalent to braided monoidal category of representations of the quantum group $U_q(\mathbb{C})$. This is a first result toward a precise relationship between the invariants of 3-manifolds arising from Chern-Simmon theory with those of Reshetikhin-Turaev. However, for a general (non abelian) lie group G the construction of the factorization algebra of observables for the quantum G-Chern-Simmons has not been described in the literature yet, and there are not concrete results relating the Wilson line operators to the category $U_q(\mathfrak{g})$-mod.

Moreover, the concrete relationship between invariants associated to $\mathbb{E}_2 \rtimes \mathbb{Z}_2$-algebras (factorization homology [AF15]) and the Reshetikhin-Turaev invariants is not yet completely understood. The author hopes that a description of Reshetikhin-Turaev invariants from the theory of $\mathbb{E}_n$-algebras, and the construction of the quantized G-Chern-Simmons factorization algebra, would lead to an analogous description of *categorified* low dimensional manifold invariants arising from $\mathbb{E}_2 \rtimes \mathbb{Z}_2$-algebras on bicategories. The author also hopes that interesting examples of the latter $\mathbb{E}_n$-algebras on bicategories should arise from categorification in representation theory.

Chapter 6

Appendix A

This Appendix contains the geometric description of the nerve of a tricategory as defined in Definition 2.3.7. We will make use of the pictures in [ner] and would like to thank the person who typed them. In the PDF version of this thesis the reader may feel free to zoom the pictures as needed, however we are afraid that in the printed version of this work many of the details and data in the figures will not be appreciated. Therefore we suggest to the reader with a printed version to look at the diagrams in [ner], or asking the author for the PDF version of the present work. We also remark that there is a slight change of notation in this appendix with respect to the main body of this thesis, this was made in order to match with the notation of this appendix with the notation in [ner].

The geometric Description of $N_3(\mathcal{T})$:

The geometric picture for the first simplices of the nerve of a tricategory:

1. For $n = 0$: The 0-simplices are determined by just one object A_0, and can be visualized as points.

2. For $n = 1$: The 1-simplices are determined by a 1-cell $f_{01} : A_0 \to A_1$, and can be visualised as lines.

3. For $n = 2$: The 2-simplices can be visualized as diagrams

$$
\begin{tikzcd}
 & A_1 \arrow[dr, "f_{12}"] \arrow[d, phantom, "\Downarrow \theta_{012}"] & \\
A_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] & {} & A_2
\end{tikzcd}
$$

4. For $n = 3$: The 3-simplices can be visualized as filled tetahedra. This can be efficiently written as a pasting diagram

$$
\begin{tikzcd}
 & A_1 \arrow[dr, "f_{12}"] \arrow[dd, "f_{13}"] & \\
A_0 \arrow[ur, "f_{01}"] \arrow[dr, "f_{03}"'] \arrow[r, phantom, "\Leftarrow \theta_{013}"] & {} & A_2 \arrow[dl, "f_{23}"] \arrow[l, phantom, "\Leftarrow \theta_{123}"] \\
 & A_3 &
\end{tikzcd}
\overset{\Gamma_{0123}}{\Rrightarrow}
\begin{tikzcd}
 & A_1 \arrow[dr, "f_{12}"] \arrow[d, phantom, "\Downarrow \theta_{012}"] & \\
A_0 \arrow[ur, "f_{01}"] \arrow[dr, "f_{03}"'] \arrow[rr, "f_{02}"] & {} \arrow[d, phantom, "\Downarrow \theta_{023}"] & A_2 \arrow[dl, "f_{23}"] \\
 & A_3 &
\end{tikzcd}
$$

5. For $n \geq 4$: Due to dimensional reasons these simplices do not have anymore an *easy* geometric description. However, they can be described by the following data:

   (a) An octahedron with 2-cells between any two composable 1-cells. In Figure 6.1 we give a 3 dimensional representation of such octahedron, together the planar views of its front and back.

   (b) A filled tetahedron for every three composable 1-cells, which are pictured in Figure 6.2. We can consider each of the 3-cells in Figure 6.2 as 2-cells in the hom bicategory between the initial vertex and final vertex of the tetrahedron. This is also done in Figure 6.2 where we omitted the information of the vertices because they can be read from the pasting diagrams of the tetahedra.

   (c) At last we have a coherence condition between the 3-cells from (b) which is depicted in Figure 6.3. In this picture the coherence is both considered as a maps of filled tetrahedra and as a pasting diagram in the bicategory $\mathcal{T}(A_0, A_4)$.

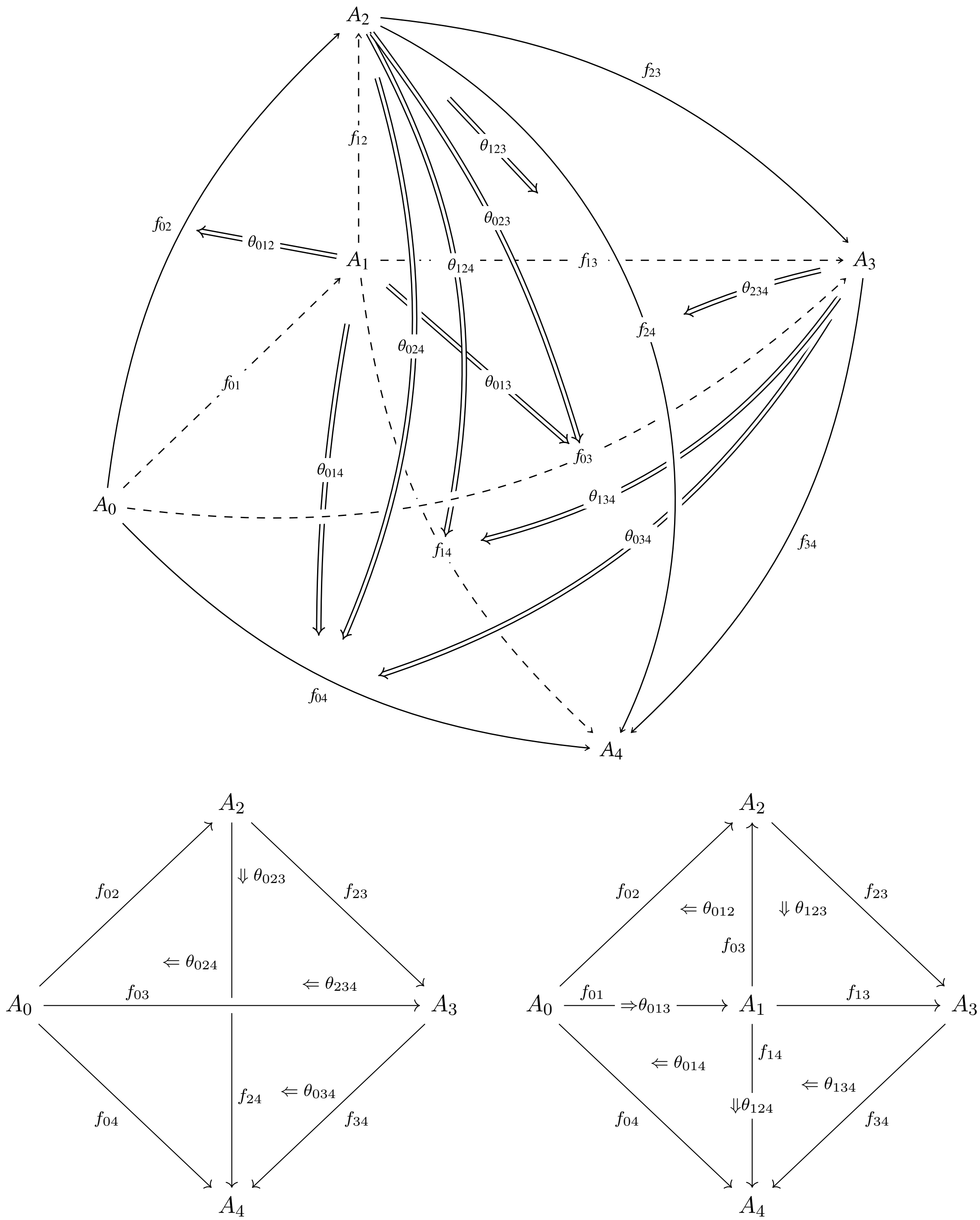


Figure 6.1: 1-cell and 2-cell data of a 4-simplex of $N_3(\mathcal{T})$ as an octahedron, together with its front and back view.

i.

$$
\begin{tikzcd}
& A_2 \arrow[dr, "f_{23}"] \arrow[dd, "f_{24}"] & \\
A_1 \arrow[ur, "f_{12}"] \arrow[dr, "f_{14}"'] \arrow[r, phantom, "\Leftarrow \theta_{124}"] & {} \arrow[r, phantom, "\Leftarrow \theta_{234}"] & A_3 \arrow[dl, "f_{34}"] \\
& A_4 &
\end{tikzcd}
\Rrightarrow_{\Gamma_{1234}}
\begin{tikzcd}
& A_2 \arrow[dr, "f_{23}"] \arrow[d, phantom, "\Downarrow \theta_{123}"] & \\
A_1 \arrow[ur, "f_{12}"] \arrow[rr, "f_{13}" description] \arrow[dr, "f_{14}"'] & {} \arrow[d, phantom, "\Downarrow \theta_{134}"] & A_3 \arrow[dl, "f_{34}"] \\
& A_4 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
{} \arrow[r, "\theta_{123}"] \arrow[d, "\theta_{234}"'] \arrow[dr, phantom, "\Uparrow \Gamma_{1234}"] & {} \arrow[d, "\theta_{034}"] \\
{} \arrow[r, "\theta_{124}"'] & {}
\end{tikzcd}
$$

ii.

$$
\begin{tikzcd}
& A_2 \arrow[dr, "f_{23}"] \arrow[dd, "f_{24}"] & \\
A_0 \arrow[ur, "f_{02}"] \arrow[dr, "f_{04}"'] \arrow[r, phantom, "\Leftarrow \theta_{024}"] & {} \arrow[r, phantom, "\Leftarrow \theta_{234}"] & A_3 \arrow[dl, "f_{34}"] \\
& A_4 &
\end{tikzcd}
\Rrightarrow_{\Gamma_{0234}}
\begin{tikzcd}
& A_2 \arrow[dr, "f_{23}"] \arrow[d, phantom, "\Downarrow \theta_{023}"] & \\
A_0 \arrow[ur, "f_{02}"] \arrow[rr, "f_{03}" description] \arrow[dr, "f_{04}"'] & {} \arrow[d, phantom, "\Downarrow \theta_{034}"] & A_3 \arrow[dl, "f_{34}"] \\
& A_4 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
{} \arrow[r, "\theta_{023}"] \arrow[d, "\theta_{234}"'] \arrow[dr, phantom, "\Uparrow \Gamma_{0234}"] & {} \arrow[d, "\theta_{034}"] \\
{} \arrow[r, "\theta_{024}"'] & {}
\end{tikzcd}
$$

iii.

$$
\begin{tikzcd}
& A_1 \arrow[dr, "f_{13}"] \arrow[dd, "f_{14}"] & \\
A_0 \arrow[ur, "f_{01}"] \arrow[dr, "f_{04}"'] \arrow[r, phantom, "\Leftarrow \theta_{014}"] & {} \arrow[r, phantom, "\Leftarrow \theta_{134}"] & A_3 \arrow[dl, "f_{34}"] \\
& A_4 &
\end{tikzcd}
\Rrightarrow_{\Gamma_{0134}}
\begin{tikzcd}
& A_1 \arrow[dr, "f_{13}"] \arrow[d, phantom, "\Downarrow \theta_{013}"] & \\
A_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{03}" description] \arrow[dr, "f_{04}"'] & {} \arrow[d, phantom, "\Downarrow \theta_{034}"] & A_3 \arrow[dl, "f_{34}"] \\
& A_4 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
{} \arrow[r, "\theta_{013}"] \arrow[d, "\theta_{134}"'] \arrow[dr, phantom, "\Uparrow \Gamma_{0134}"] & {} \arrow[d, "\theta_{024}"] \\
{} \arrow[r, "\theta_{014}"'] & {}
\end{tikzcd}
$$

iv.

$$\begin{tikzcd}
 & A_1 \arrow[dr, "f_{12}"] \arrow[dd, "f_{14}"] & & & & A_1 \arrow[dr, "f_{12}"] & \\
A_0 \arrow[ur, "f_{01}"] \arrow[dr, "f_{04}"'] \arrow[r, phantom, "\Leftarrow \theta_{014}"] & {} \arrow[r, phantom, "\Leftarrow \theta_{124}"] & A_2 \arrow[dl, "f_{24}"] \arrow[r, phantom, "\Rrightarrow \Gamma_{0124}"] & {} & A_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"] \arrow[dr, "f_{04}"'] & \Downarrow \theta_{012} \quad \Downarrow \theta_{024} & A_2 \arrow[dl, "f_{24}"] \\
 & A_4 & & & & A_4 &
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[rr, "\theta_{012}"] \arrow[d, "\theta_{124}"'] & \Uparrow \Gamma_{0124} & {} \arrow[d, "\theta_{023}"] \\
{} \arrow[rr, "\theta_{014}"'] & & {}
\end{tikzcd}$$

v.

$$\begin{tikzcd}
 & A_1 \arrow[dr, "f_{12}"] \arrow[dd, "f_{13}"] & & & & A_1 \arrow[dr, "f_{12}"] & \\
A_0 \arrow[ur, "f_{01}"] \arrow[dr, "f_{04}"'] \arrow[r, phantom, "\Leftarrow \theta_{013}"] & {} \arrow[r, phantom, "\Leftarrow \theta_{123}"] & A_2 \arrow[dl, "f_{23}"] \arrow[r, phantom, "\Rrightarrow \Gamma_{0234}"] & {} & A_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"] \arrow[dr, "f_{04}"'] & \Downarrow \theta_{012} \quad \Downarrow \theta_{023} & A_2 \arrow[dl, "f_{23}"] \\
 & A_3 & & & & A_3 &
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[rr, "\theta_{012}"] \arrow[d, "\theta_{123}"'] & \Uparrow \Gamma_{0234} & {} \arrow[d, "\theta_{134}"] \\
{} \arrow[rr, "\theta_{013}"'] & & {}
\end{tikzcd}$$

Figure 6.2: 3-cell data of a 4-simplex of $N_3(\mathcal{T})$ as filled tetahedra and as 2-cells in some hom bicategories.

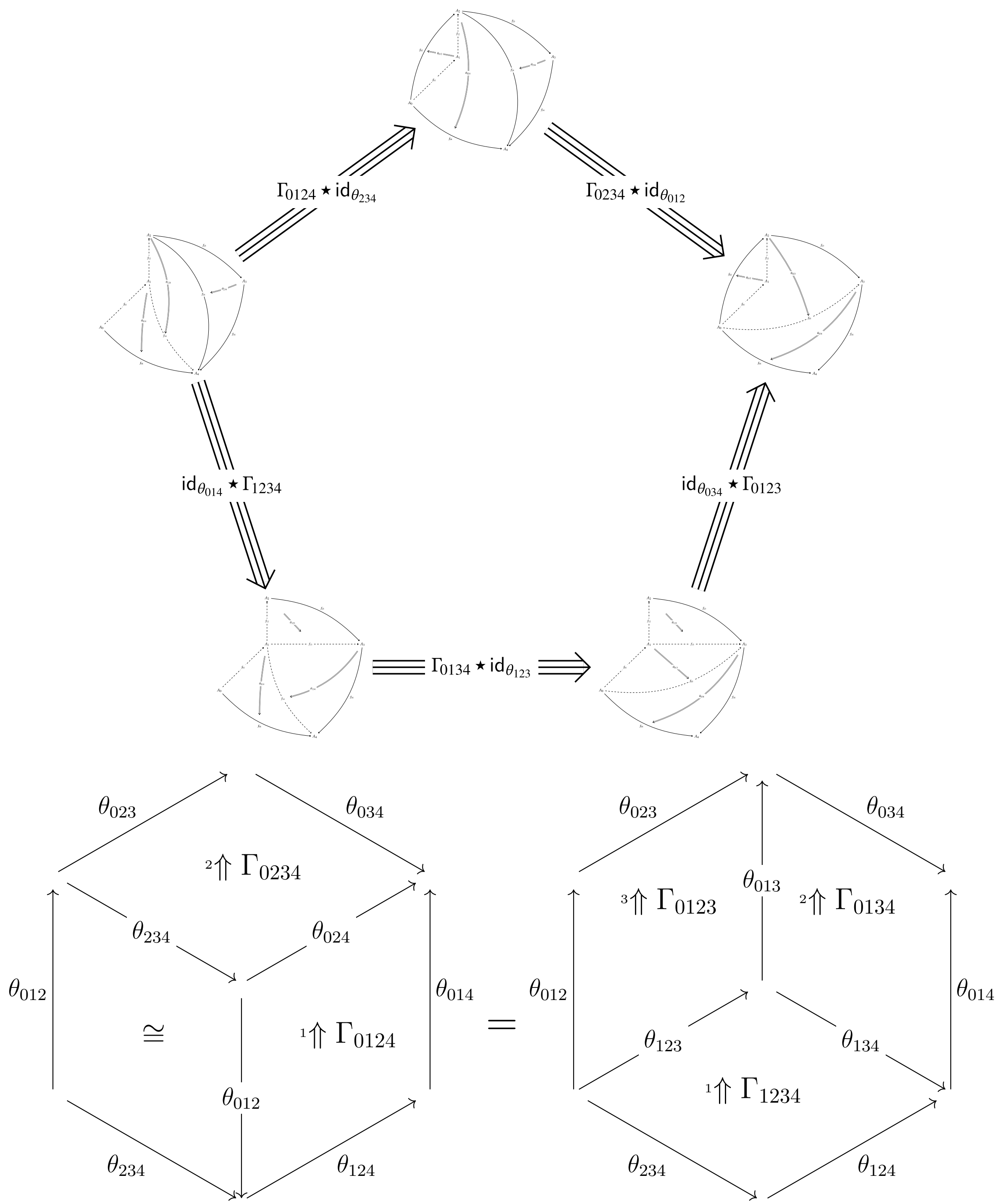


Figure 6.3: Coherence condition for the 3-cell data of a 4-simplex in $N_3(\mathcal{T})$. As a diagram of tetahedra and as a pasting diagram of 2-cells in the bicategory $\mathcal{T}(A_0, A_4)$.

Chapter 7

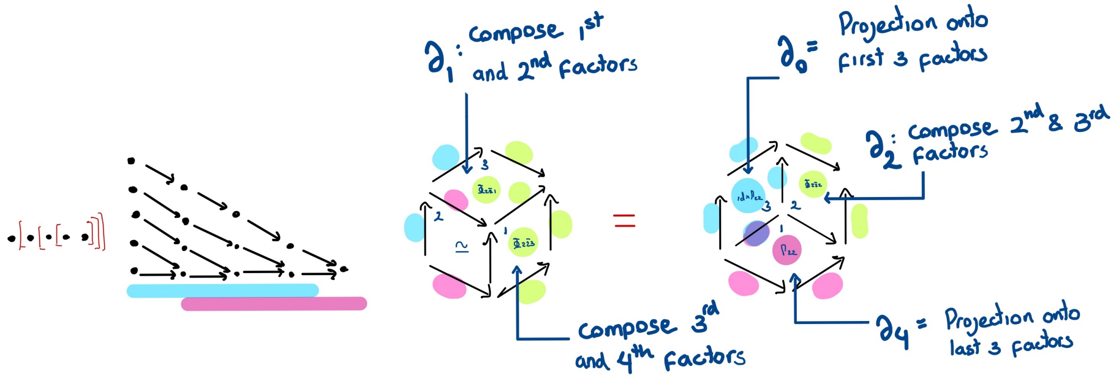
Appendix B

This appendix contains the explicit verification that the images of the trees appearing in the proofs of Theorem 0.0.4, Theorem 0.0.5 and Theorem 0.0.6 are the indeed the polytopes described in these proofs. These are given by figures relation each possible nondegenerate simples in $\mathbb{A}$ss or $\mathbb{A}$ss $\times$ $\mathbb{A}$ss in degree less or equal to 4. In the PDF version of this thesis the reader may feel free to zoom the pictures as needed, however we are afraid that in the printed version of this work many of the details and data in the figures will not be appreciated. Therefore we either suggest to the reader with a printed version to ask the author for the PDF version of the present work. Before presenting the computations we explain the notations used in the below pictures.

- Recall that each nondegenerate tree of length 4 give rise to a cube in the bicategory of pseudofunctors from $\mathcal{B}^5$ to $\mathcal{B}$ (see the proof of Theorem 0.0.5)

  As explained in theorem 0.0.5, each nondegenerate tree of length 4 give rise to a cube in the bicategory of pseudofunctors from $\mathcal{B}^5$ to $\mathcal{B}$ (see the proof of Theorem 0.0.5). These cubes have internal and external faces. We want to check that the external faces match the necessary morphism to define the coherence conditions on each of the theorems. While the internal faces should face each other, in order to conclude that the external faces indeed describe a equality of pasting diagrams. In the below pictures the internal faces are colored in yellow, and the pairs of faces that face each other are and numbered in orange. The above calculation check that indeed the internal faces match in the coherent way that allows to conclude the desired commutativity in each case.

- For the reader interested in checking, or repeating, the calculations we add the following example which graphically explains the relationship between the faces maps of the 4-simplices in the quasioperad $\mathbb{A}$ss (which are represented by trees of length 4) and the faces of the 4-simplices in $\mathbb{G}$ray.



Thus to read the 2-morphism on the cube associated to the 4-simplex in **G**ray it is enough to read the length 3 trees given by the faces of the 4-simplex, which should have been already assigned in the description of the map $\mathbb{A}\text{ss} \to \mathbb{G}\text{ray}$ on 3-simplices.

- There exist cases where the same edge of a polytope may be represented by more than one tree. In these cases the cube given in the decomposition of the polytopes below are made from smaller cubes pasted together. We colored the faces of the these smaller cubes in a red/dark-orange color and numbered the corresponding pairs in red. Knowing how faces match to each other allows the reader to figure how the *smaller* cubes build into the cube appearing in the figures.

$\mathbb{E}_1$-algebras in $\mathbb{G}$ray

In the proof of Theorem 0.0.5 we gave an intuition of the decomposition of the 4-Associahedron in term of the simplicices arising from nondegenerate trees of length 4. Below we present for each length 4 tree the corresponding pasting diagram representing its image in $\mathbb{G}$ray and verify they match together as the 4-associahedron.

$\mathbb{E}_2$-algebras in $\mathbb{C}$at and $\mathbb{G}$ray

In the proofs of Theorem and 0.0.4 and Theorem 0.0.6 we presented the diagrams and polytopes decomposition that gave the description of $\mathbb{E}_2$-algebras on $\mathbb{C}$at and $\mathbb{G}$ray. Here to each non-degenerate tree of lengths 3 and 4 we show the corresponding pasting diagram and verify they match together in the polytopes of Chapter 4. Moreover, to find the corresponding morphism from a product of trees we used the discussion in Remark 4.1.11 in the main body of the work. In the case of $\mathbb{E}_2$-algebras on $\mathbb{C}$at we need to truncate the morphisms appearing in the below picture to arrive at our description, i.e. we consider the modifications Γ's and Δ's in the below pictures to be equalities.

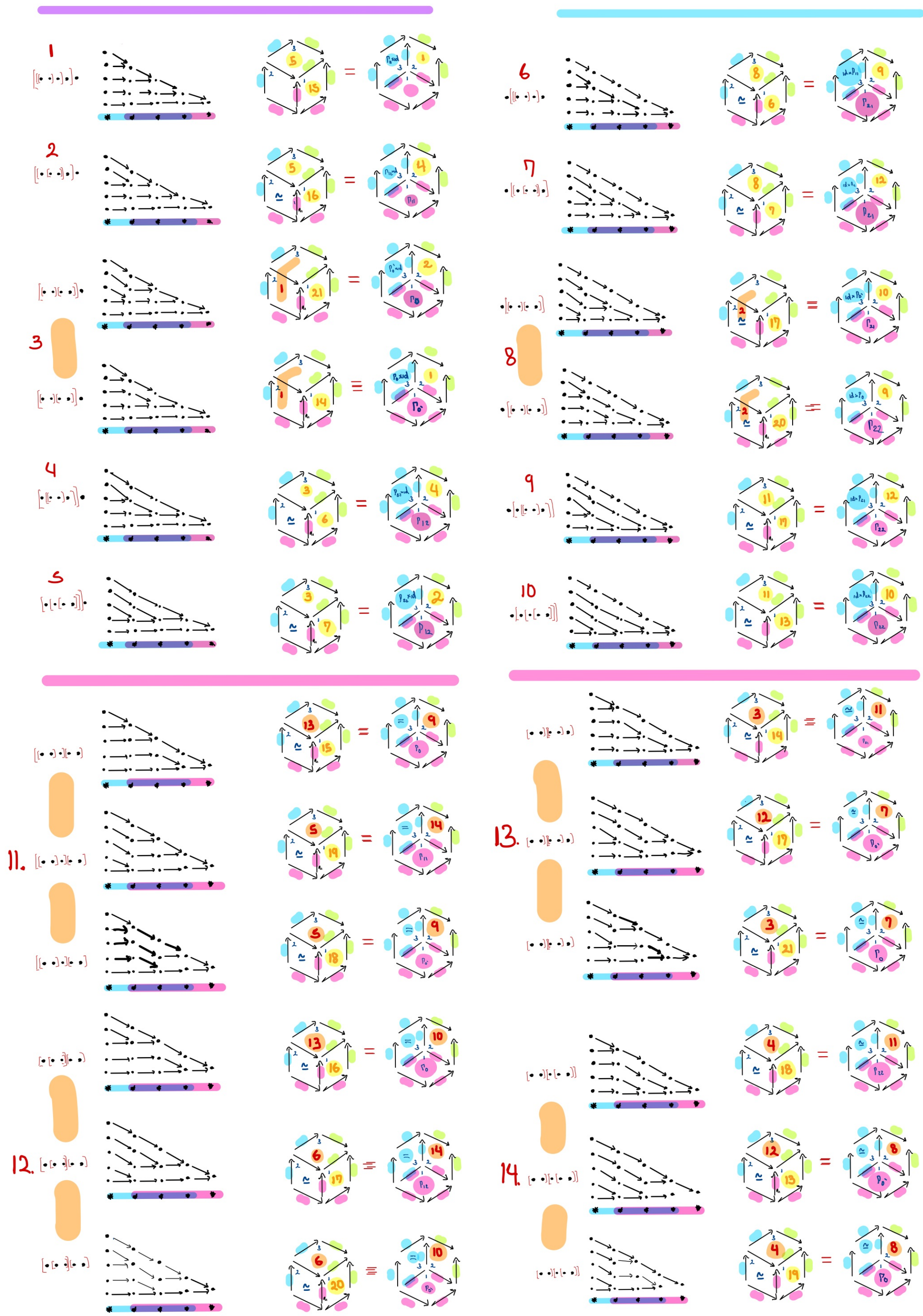


Figure 7.1: Images of all nondegenerate 4-simplices for a $\mathbb{E}_1$-algebra in $\mathbb{G}$ray.

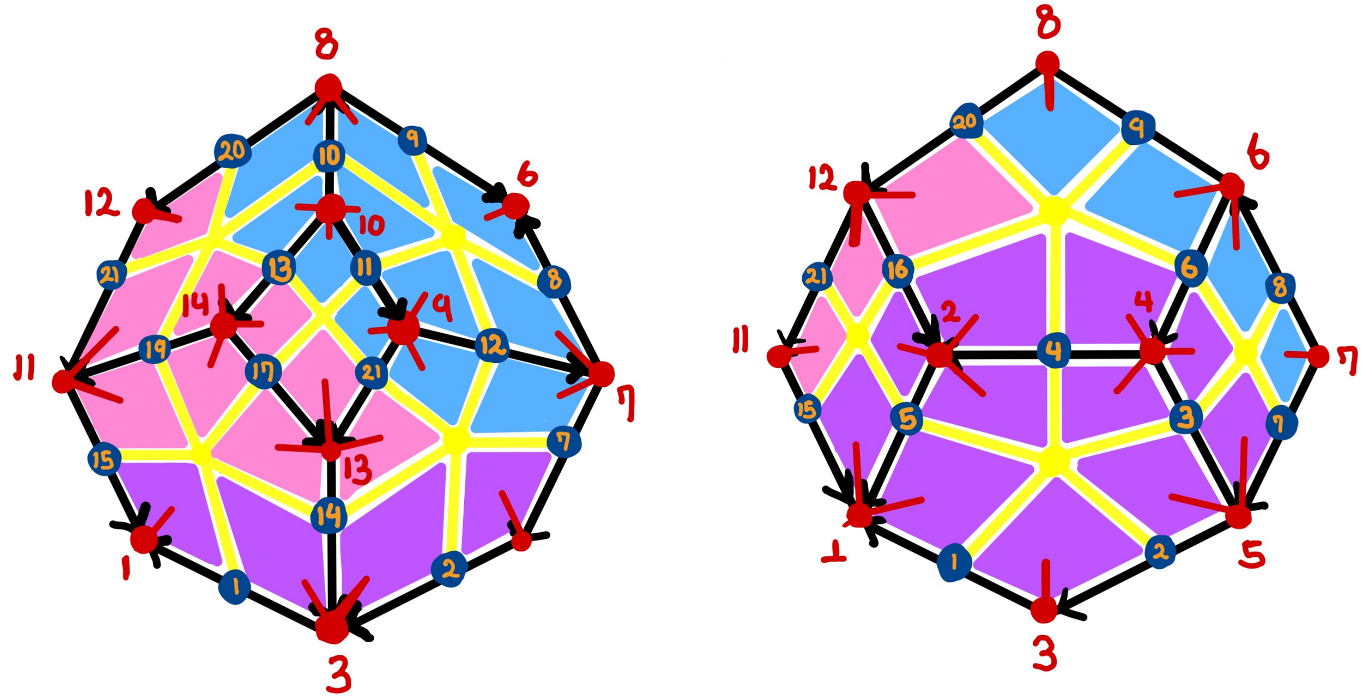


Figure 7.2: Decomposition of the 4-associahedron into cubes. Each cube is marked by a red number corresponding to a unique bracketing and each pair of internal faces facing each other is marked by a blue dot with the appropriate orange label determining the internal faces.

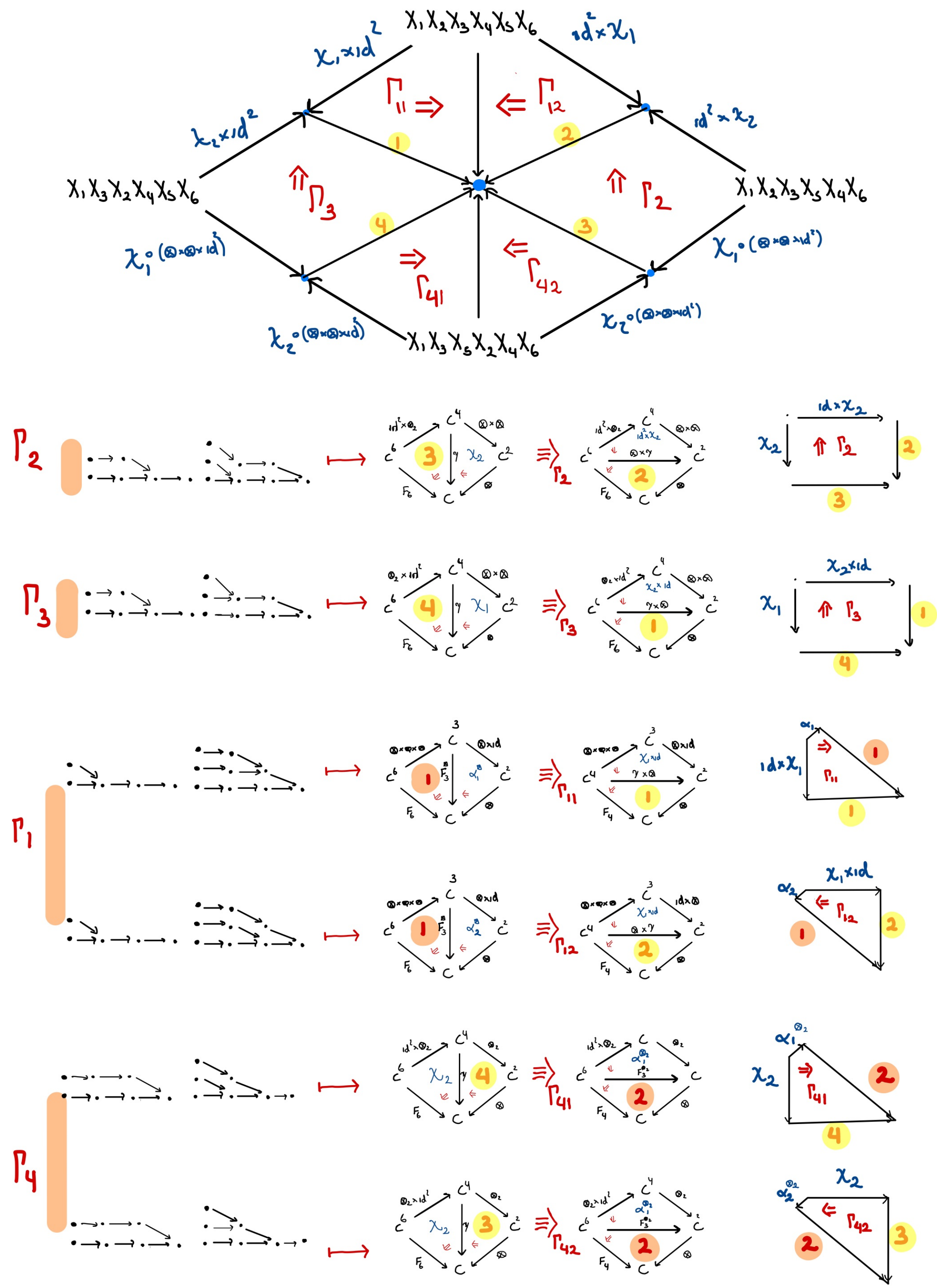


Figure 7.3: Images of the nondegenerate 3-simplices of $\mathbb{A}\text{ss} \times \mathbb{A}\text{ss}$ in $\mathbb{G}\text{ray}$.

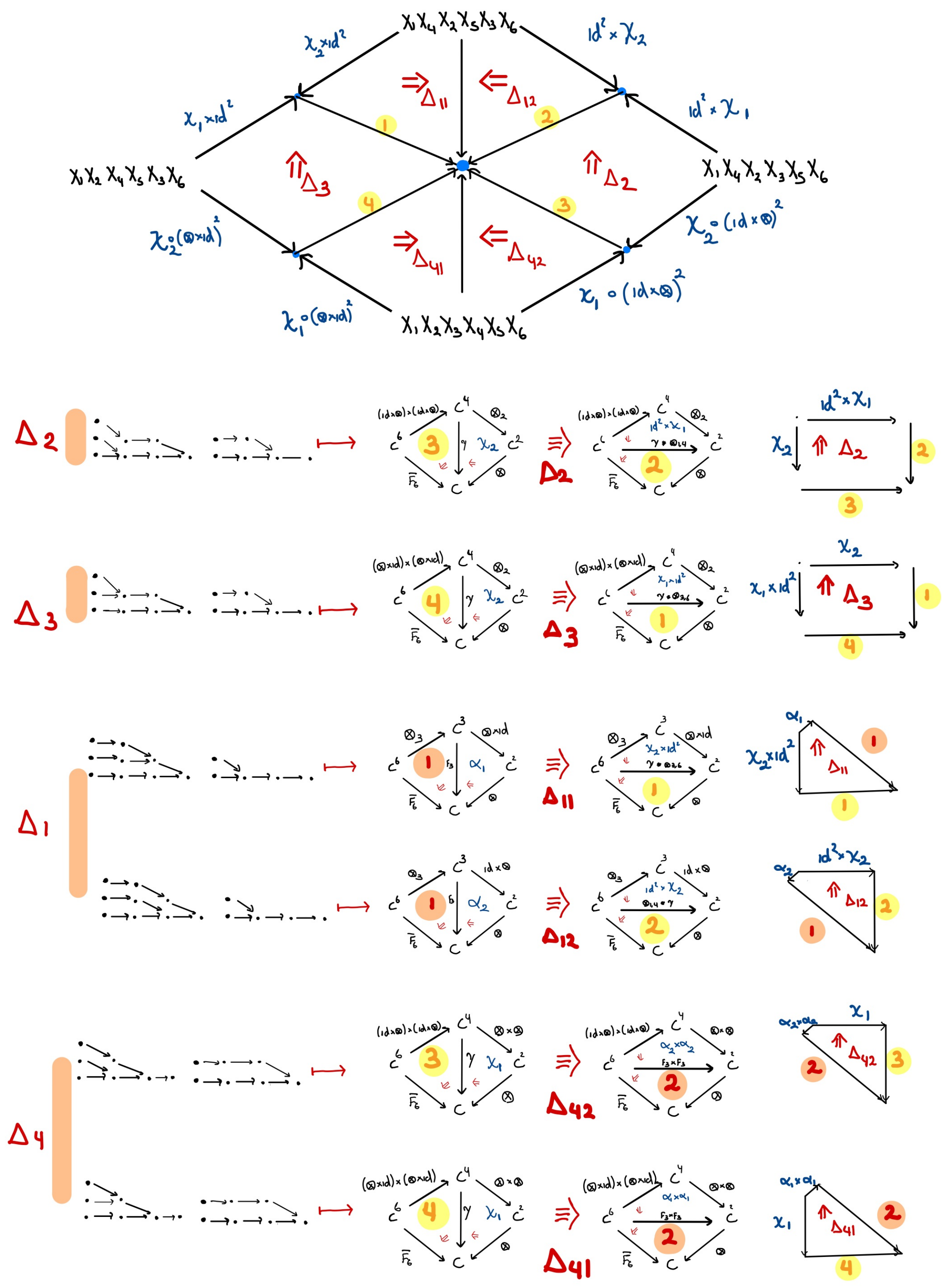


Figure 7.4: Images of the nondegenerate 3-simplices of $\mathbb{A}\mathrm{ss} \times \mathbb{A}\mathrm{ss}$ in $\mathbb{G}\mathrm{ray}$.

Figure 7.5: Images of the nondegenerate 4-simplices of $\mathbb{A}\mathrm{ss} \times \mathbb{A}\mathrm{ss}$ in $\mathbb{G}\mathrm{ray}$.

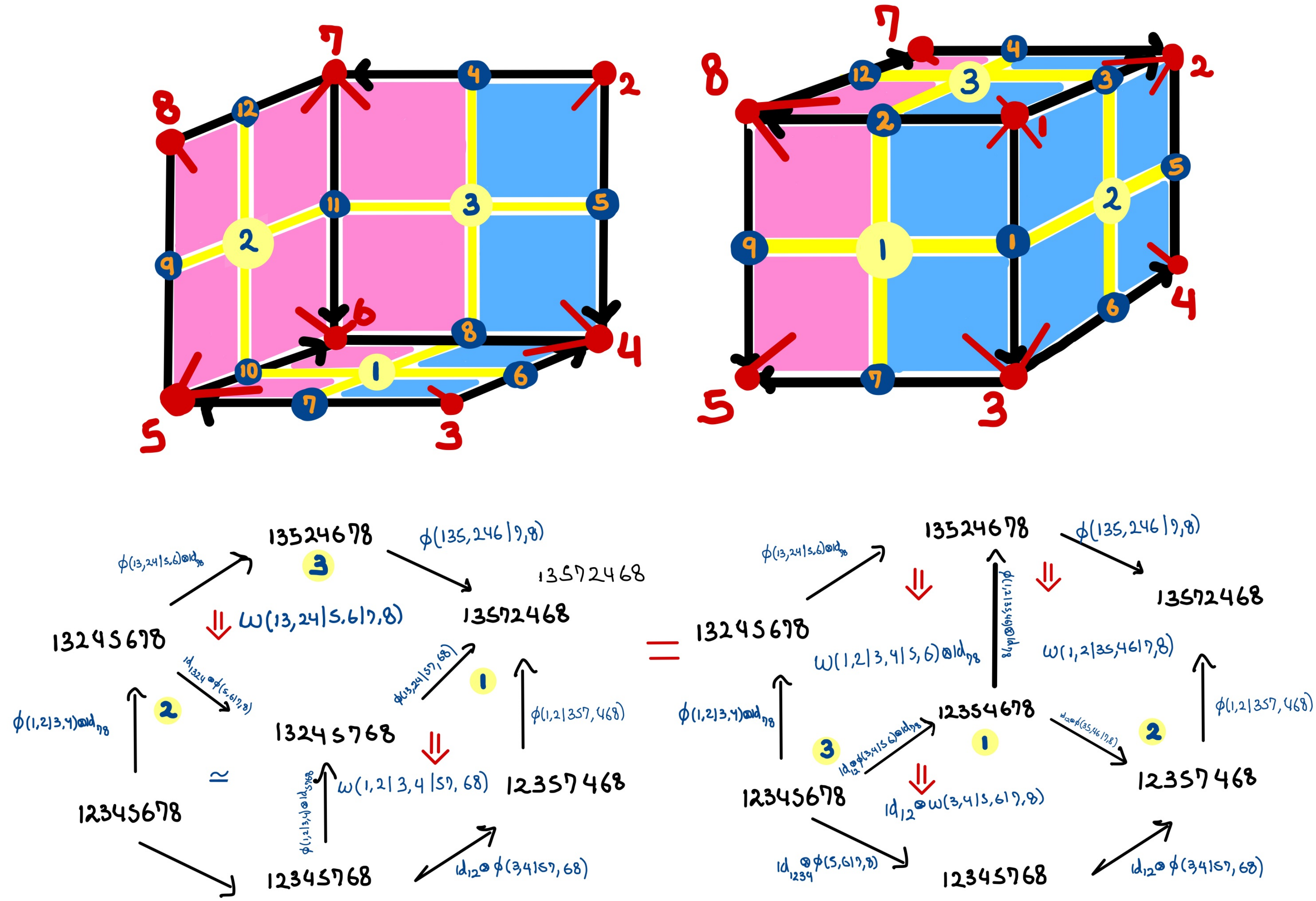


Figure 7.6: Decomposition of the cube into smaller cubes. Together with the associated pasting diagrams.

Figure 7.7: Images of the nondegenerate 4-simplices of $\mathbb{A}\mathrm{ss} \times \mathbb{A}\mathrm{ss}$ in $\mathbb{G}\mathrm{ray}$.

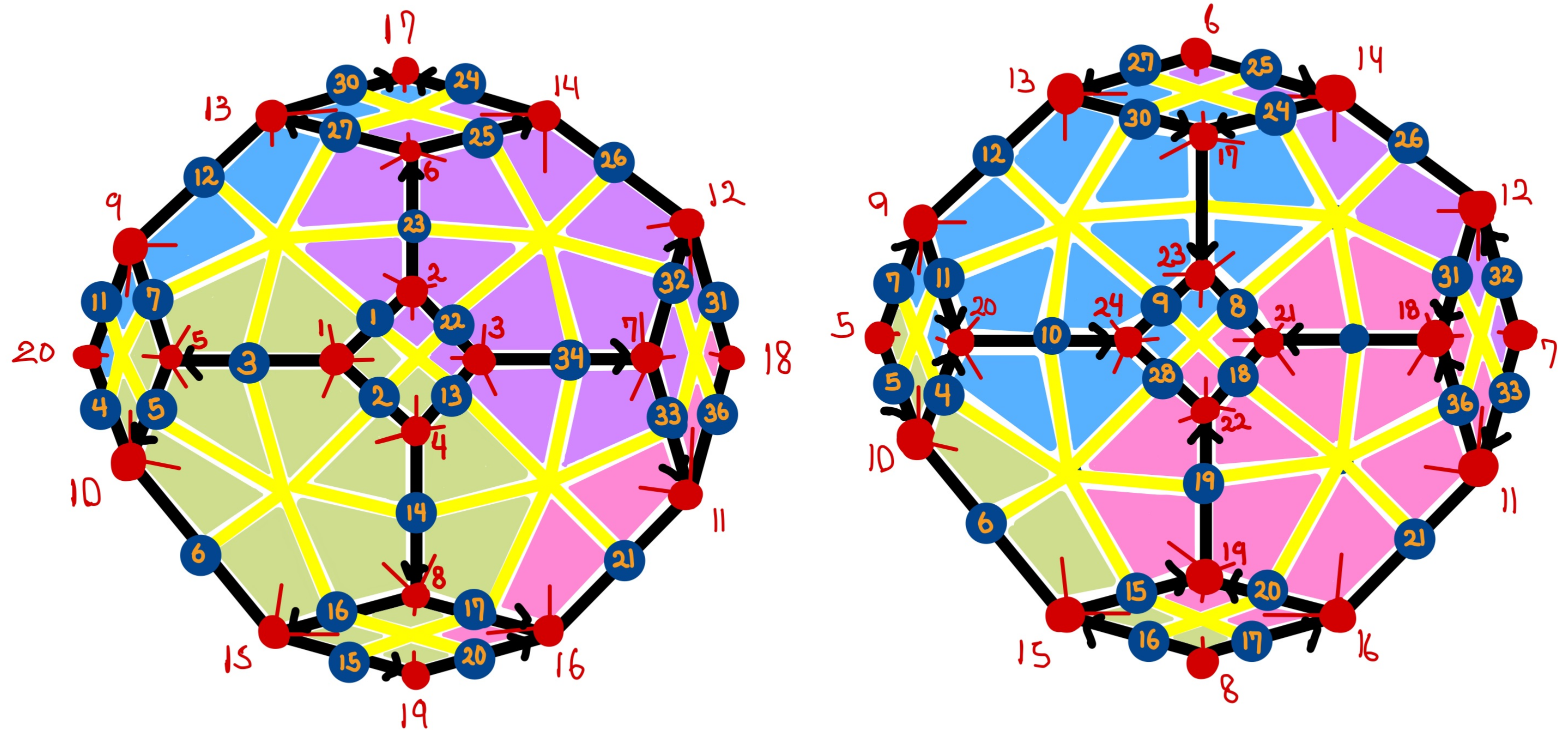


Figure 7.8: Decomposition of the polytope appearing in $\mathbb{E}_2$-agebras on $\mathbb{G}\mathrm{ray}$ into smaller cubes. Each cube is marked by a red number corresponding to a unique bracketing and each pair of internal faces facing each other is marked by a blue dot with the appropriate orange label determining the internal faces.

Chapter 8

Appendix C

This Appendix is devoted to the description of Cartesian symmetric monoidal quasicategories. Most of the material in this appendix can be found in [Lur09b, §2.4]. The main goal of this Appendix is to give detailed proofs of Proposition 3.3.6 and Proposition 4.1.10 used in the main body of this thesis.

Kan Extensions

The proofs of Proposition 3.3.6 and Proposition 4.1.10 will use the concept of Kan extension in quasicategories, here we introduce them and describe their main properties. The reader already familiar with Kan extensions may skip this subsection. Let $\mathcal{C}$ be a quasicategory, and let $\mathcal{C}^0$ be a full subquasicategory. If $C \in \mathcal{C}_0$ is an object we will denote by $\mathcal{C}^0_{C/}$ the fullsubquasicategory of $\mathcal{C}_{C/}$ spanned by the 1-simplices $C \to C'$ where $C' \in \mathcal{C}^0$.

Definition 8.0.1. Let $\mathcal{C}$ be a quasicategory and let $\mathcal{C}^0$ be a full subquasicategory. And let the following be a commutative diagram of simplicial sets

$$\begin{tikzcd} \mathcal{C}^0 \arrow[r, "F_0"] \arrow[d] & \mathcal{D} \\ \mathcal{C} \arrow[ur, "F"'] & \end{tikzcd}$$

For every $c \in C$ the functor F determines a composition $\mathcal{C}^0_{C/} \to \mathcal{C}_{C/} \to \mathcal{D}_{F(C)/}$, which can be considered as a map $\Delta^0 \star \mathcal{C}^0_{C/} \to \mathcal{D}$ extending $F_C : \mathcal{C}^0_{C/} \to \mathcal{D}$. We will say that F is a **Kan extension at** C if the induced diagram

$$\begin{tikzcd} \mathcal{C}^0_{C/} \arrow[r, "F_C"] \arrow[d] & \mathcal{D} \\ \Delta^0 \star \mathcal{C}^0_{C/} \arrow[ur, "F"'] & \end{tikzcd}$$

is a limit diagram, that is $F(C)$ is a limit of F_C. We say F is a **Kan extension** if it is a Kan extension at C for every $C \in \mathcal{C}$.

Proposition 8.0.2. In the setup of Definition 8.0.1: F is a right Kan extension of $F|_{\mathcal{C}^0}$ at C for every $C \in \mathcal{C}^0$.

Proof. See [Lur09a, Remark 4.3.2.3.] $\square$

Proposition 8.0.3. In the setup of Definition 8.0.1: If C and C' are equivalent objects in $\mathcal{C}$, then F is a right Kan extension of $F|_{\mathcal{C}^0}$ at C if and only if it is a right Kan extension an C'.

Proof. See [Lur09b, Lemma 4.3.2.5.]. □

Proposition 8.0.4. Let $\mathcal{C}$ be a quasicategory, and let $\mathcal{C}^0$ be a full subquasicategory.

- Let $\mathrm{Kan}(\mathcal{C},\mathcal{D}) \subset \underline{\mathrm{sSets}}(\mathcal{C},\mathcal{D})$ be the fullsubquasicategory spanned by the simplicial sets maps $F : \mathcal{C} \to \mathcal{D}$ that are right Kan extensions of $F|_{\mathcal{C}^0} : \mathcal{C}^0 \to \mathcal{D}$.
- Let $\mathrm{PreKan}(\mathcal{C}^0,\mathcal{D}) \subset \underline{\mathrm{sSets}}(\mathcal{C}^0,\mathcal{D})$ be the fullsubquasicategory spanned by the simplicial sets maps $F : \mathcal{C}^0 \to \mathcal{D}$ such that for each object $C \in \mathcal{C}_0$ the induced diagram $\mathcal{C}^0_{/C} \to \mathcal{D}$ has a limit.

Then the restriction $\mathcal{C}^0 \to \mathcal{C}$ induces an equivalence of categories

$$\mathrm{Kan}(\mathcal{C},\mathcal{D}) \to \mathrm{PreKan}(\mathcal{C}^0,\mathcal{D}).$$

Proof. See [Lur09b, Proposition 4.3.2.15] □

Symmetric Monoidal Cartesian Quasicategories

Definition 8.0.5. The category $\mathrm{Fin}_*^\times$ is the category with:

1. Objects given by tuples $(\langle n\rangle, S)$, where $\langle n\rangle$ is an object of Fin_* and $S \subset \langle n\rangle^\circ$, where $\langle n\rangle^\circ = \langle n\rangle \setminus \{*\}$.
2. Morphisms from $(\langle n\rangle, S) \to (\langle m\rangle, S')$ are given by a map $\alpha : \langle n\rangle \to \langle m\rangle$ such that $\alpha^{-1}S' \subset S$.

Remark 8.0.6.

- There is a Cartesian fibration $U : \mathrm{Fin}_*^\times \to \mathrm{Fin}_*$ defined on objects by $(\langle n\rangle, S) \mapsto \langle n\rangle$ and on morphism by $\alpha \mapsto \alpha$. The Cartesian lifts can be described as follows: for every $\alpha : \langle n\rangle \to \langle m\rangle$ and $S \subset \langle m\rangle^\circ$, there is a Cartesian lift determined by $\alpha^! : (\langle n\rangle, \alpha^{-1}S) \to (\langle m\rangle, S)$.
- Moreover, there is a canonical section $s : \mathrm{Fin}_* \to \mathrm{Fin}_*^\times$ given on objects by $\langle n\rangle \mapsto (\langle n\rangle, \langle n\rangle^\circ)$ and on morphisms by $\alpha \mapsto \alpha$.

Let $\mathcal{C}$ be a quasicategory, then we define $\mathcal{C}^\Pi$ to be simplicial set over $N(\mathrm{Fin}_*)$ by the property

$$\mathrm{sSets}_{N(\mathrm{Fin}_*)}(K, \mathcal{C}^\Pi) \cong \mathrm{sSets}(K \times_{N(\mathrm{Fin}_*)} N(\mathrm{Fin}_*^\times), \mathcal{C}).$$

Remark 8.0.7.

- For ever simplicial set K the section $s : \mathrm{Fin}_* \to \mathrm{Fin}_*^\times$ induces a map of sets

$$\mathrm{sSets}(K \times_{N(\mathrm{Fin}_*)} N(\mathrm{Fin}_*^\times), \mathcal{C}) \to \mathrm{sSets}(K \times_{N(\mathrm{Fin}_*)} N(\mathrm{Fin}_*), \mathcal{C}),$$

  which by the Yoneda lemma induces a map of simplicial sets $\pi : \mathcal{C}^\Pi \to \mathcal{C}$.
- The fact that all the above simplicial sets are quasicategories is nontrivial and is consequence of [Lur09b, Corollary 3.2.2.12]

We can characterize the 0-simplices and 1-simplices of $\mathcal{C}^\times$ in the following way. Let $i = 0, 1$ and let $\Delta^i \to N(\mathrm{Fin}_*)$ be an i-simplex over $N(\mathrm{Fin}_*)$. By definition we have

$$\mathrm{sSet}_{N(\mathrm{Fin}_*)}(\Delta^i, \mathcal{C}^\Pi) \cong \mathrm{sSet}(\Delta^i \times_{N(\mathrm{Fin}_*)} N(\mathrm{Fin}_*^\times), \mathcal{C}).$$

Since the nerve functor preserve colimits and is fully faithful we have

$$\Delta^0 \times_{N(\mathrm{Fin}_*)} N(\mathrm{Fin}_*^\times) = N(\Delta^i \times_{\mathrm{Fin}_*} \mathrm{Fin}_*^\times).$$

Thus

$$\mathrm{sSet}_{N(\mathrm{Fin}_*)}(\Delta^i, \mathcal{C}^\Pi) \cong \mathrm{sSet}(N(\Delta^i \times_{N(\mathrm{Fin}_*)} \mathrm{Fin}_*^\times), \mathcal{C}).$$

Note that the argument works just for $i = 0, 1$ since these are the dimension in which Δ^i is equivalent to the nerve of a category. A detailed description of the i-simplices follows from characterizing of the categories $\Delta^i \times_{\mathrm{Fin}_*} \mathrm{Fin}_*^\times$.

Lemma 8.0.8.

1. Let P_n be the category whose objects are subsets of $\langle n \rangle^\circ$, and with morphims $I \to J$ if and only if $J \subset I$. Then the 0-simplices of $\mathcal{C}^\Pi$ over $N(\mathrm{Fin}_*)$ correspond to maps of simplicial sets $N(P_n) \to \mathcal{C}$ for some $n \in \mathbb{N}$.

2. Let $\Delta^1 \to \mathrm{Fin}_*$ be a map of simplicial sets determined by a map $\alpha : \langle n \rangle \to \langle m \rangle$ in Fin_*. Let P_α be the category whose objects are subsets of $\langle m \rangle^\circ$, and with morphism $(\langle n \rangle, S') \to (\langle m \rangle, S)$ with $S' \subset \alpha^{-1}S$. Then the 1-simplices of $\mathcal{C}^\Pi$ over $N(\mathrm{Fin}_*)$ correspond to maps of simplicial sets $N(P_\alpha) \to \mathcal{C}$ for some $n \in \mathbb{N}$. In particular, for every morphism in P_α there is an induced 1-simplex in $\mathcal{C}$.

Proof.

1. Let $\Delta^0 \to \mathrm{Fin}_*$ be a map of simplicial sets determined by the object $\langle n \rangle$ in Fin_*. It easy to see that $\Delta^0 \times_{\mathrm{Fin}_*} \mathrm{Fin}_*^\times$ and P_n are equivalent categories, therefore we conclude the desired result.

2. It easy to see that $\Delta^1 \times_{\mathrm{Fin}_*} \mathrm{Fin}_*^\times$ and P_α are equivalent categories, therefore we conclude the desired result.

$\square$

Definition 8.0.9. Let $\mathcal{C}$ be a quasicategory with finite limits. The **symmetric monoidal Cartesian quasicategory** $\mathcal{C}^\times$ is the full subquasicategory of $\mathcal{C}^\Pi$ spanned by the objects corresponding to functors $F : N(P_{\langle n \rangle}) \to \mathcal{C}$ such that

$$\prod F\rho_j : F(S) \to F(\{j\})$$

is an equivalence. Here ρ_i denotes the unique morphism in $P_{|S|}$ determined by the inclusion $\{j\} \subset S$.

Proposition 8.0.10.

1. The projection $p : \mathcal{C}^\times \to N(\mathrm{Fin}_*)$ is a coCartesian fibration, i.e. $\mathcal{C}^\times$ is a symmetric monoidal quasicategory.

2. A 1-simplex F over $\alpha : \langle n \rangle \to \langle m \rangle$ is coCartesian in $\mathcal{C}^\times$ if and only if, for every $S \subset \langle m \rangle^\circ$ the induced 1-simplex $F(\langle n \rangle, \alpha^{-1}S) \to F(\langle m \rangle, S)$ is an equivalence in $\mathcal{C}$. Here induced refers to Lemma 8.0.8.

3. The map $\pi : \mathcal{C}^{\times} \to \mathcal{C}$ induced by the section $s : \mathrm{Fin}_* \to \mathrm{Fin}_*^{\times}$ is a Lax Cartesian morphism.

Proof. See [Lur09a, Proposition 2.4.1.5]. □

We can now state the two main proposition of this appendix:

Proposition 8.0.11. Let $\mathcal{O}^{\otimes}$ be an quasioperad and $\mathcal{D}$ a quasicategory with finite products. The composition with $\pi : \mathcal{C}^{\times} \to \mathcal{C}$ induces an equivalence of categories

$$\pi^* : \mathrm{Alg}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\times}) \to \underline{\mathrm{sSets}}^{\mathrm{Lax}}(\mathcal{O}^{\otimes}, \mathcal{C}).$$

Proposition 8.0.12. Let $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ be an quasioperads and $\mathcal{D}$ a quasicategory with finite products. The composition with $\pi : \mathcal{C}^{\times} \to \mathcal{C}$ induces an equivalence of categories

$$\mathrm{BiFunc}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}; \mathcal{C}^{\times}) \cong \underline{\mathrm{sSets}}^{\mathrm{BiLax}}(\mathcal{O}^{\otimes}, \mathcal{C}).$$

The reminder of this appendix is devoted to the proofs of Proposition 8.0.11 and Proposition 8.0.12. The argument presented here was originally given in [Lur09b, Proposition 2.4.1.7] as a proof of Proposition 8.0.11. We write it here again for two reasons: first for the convenience of the reader; second, and most importantly, to make clear the required modifications of the argument for the proof of Proposition 8.0.12.

Lemma 8.0.13. Let $\mathcal{C}^0 \subset \mathcal{C}$ be a full subquasicategory and let $C \in \mathcal{C}_0$ be an object. Assume that $\mathcal{C}_{C/}$ has an initial object $\eta : C \to I$, where the vertex I is an object of $\mathcal{C}^0$. Then any map of simplicial sets $F^0 : \mathcal{C}^0 \to \mathcal{D}$ admits a right Kan extension at C,and an arbitrary map of simplicial sets $F : \mathcal{C} \to \mathcal{D}$ is a right Kan extension of $F|_{\mathcal{C}^0}$ at C, if and only if $F(\eta)$ is a equivalence in $\mathcal{D}$.

Proof. By [Rez21, Proposition 33.8] the edge $\eta : C \to I$ is an equivalence in $\mathcal{C}$. By 8.0.2 every map of simplicial sets $F^0 : \mathcal{C}^0 \to \mathcal{D}$ admits a right Kan extension at I (since $I \in C^0$). Then the statement follows follows from Proposition 8.0.3 and the fact that $\eta : C \to I$ is an equivalence. □

Proof of Proposition 8.0.11

In what follows we will consider $\mathcal{O}^{\otimes}$ as a full subquasicategory of $\mathcal{O}^{\otimes} \times_{N(Fin_*)} N(\mathrm{Fin}_*^{\times})$ spanned by the objects of the form $(C, \langle n \rangle^{\circ})$ for $C \in \mathcal{O}^{\otimes}_{\langle n \rangle}$.

Lemma 8.0.14. With the assumptions of Proposition 8.0.11. Let (C, S) be an object of $\mathcal{O}^{\otimes} \times_{N(\mathrm{Fin}_*)} \mathrm{Fin}_*^{\times}$, thus $C \in \mathcal{O}^{\otimes}_{\langle n \rangle}$ and $S \subset \langle n \rangle^{\circ}$. Let $\alpha : \langle n \rangle \to \langle S \rangle$ be the map defined by

$$\alpha(j) = \begin{cases} j & \text{if } j \in S \\ * & \text{otherwise,} \end{cases}$$

and let $C \to C'$ and $(\alpha^{-1}S) \to S$ be (co)Cartesian lifts of α in $\mathcal{O}^{\otimes}$ and $\mathrm{Fin}_*^{\times}$ respectively. Then the induced 1-simplex $\widehat{\alpha} : (C, S) \to (C', S)$ is an initial object of $(\mathcal{O}^{\otimes} \times_{N(\mathrm{Fin}_*)} \mathrm{Fin}_*^{\times})_{(C,S)/}$, where (C', S) is an object of $\mathcal{O}^{\otimes}$ (since C' is an object of $\mathcal{O}^{\otimes}_{\langle S \rangle}$).

Proof. First of all is not hard to see that $\widehat{\alpha}$ is an initial object in the category ${\mathrm{Fin}_*^{\times}}_{(\langle n \rangle, S)/}$. Indeed, for another object $\eta : (\langle n \rangle, S) \to (\langle m \rangle, S')$ in ${\mathrm{Fin}_*^{\times}}_{(\langle n \rangle, S)/}$ there exist a unique map

$s : \alpha \to \eta$ given by

$$\begin{tikzcd} (\langle n\rangle, S) \arrow[r, "\alpha"] \arrow[d, "\eta"'] & (\langle S\rangle, S) \arrow[dl, dashed, "s"] \\ (\langle m\rangle, S') & \end{tikzcd} \qquad s(j) = \eta(j), \text{ for } j \in S \subset \langle n\rangle.$$

By 2-coskeletality of the nerve of a category, it easy to see that this implies that α is also an initial object in $N((\mathrm{Fin}_*^\times)_{(\langle n\rangle,S)}/) = N(\mathrm{Fin}_*^\times)_{(\langle n\rangle,S)/}$.

Now, unrolling definitions we see that an object f in $(\mathcal{O}^\otimes \times_{N(\mathrm{Fin}_*)} \mathrm{Fin}_*^\times)_{(C,S)/}$ is initial if there exists a lift for every diagram

$$\begin{tikzcd} \Delta^0 \star \Delta^0 \arrow[r] \arrow[rr, bend left, "f"] & \Delta^0 \star \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{O}^\otimes \times_{N(\mathrm{Fin}_*)} N(\mathrm{Fin}_*^\times) \\ & \Delta^0 \star \Delta^n \arrow[ur, dotted] & \end{tikzcd},$$

which using the facts that $\Delta^0 \star \Delta^n = \Delta^{n+1}$ and $\Delta^0 \star \partial\Delta^n = \Lambda_0^{n+1}$ is equivalent to

$$\begin{tikzcd} \Delta^{(0,1)} \arrow[r] \arrow[rr, bend left, "f"] & \Lambda_0^n \arrow[r] \arrow[d] & \mathcal{O}^\otimes \times_{N(\mathrm{Fin}_*)} N(\mathrm{Fin}_*^\times) \\ & \Delta^n \arrow[ur, dotted] & \end{tikzcd}.$$

By the universal property of the pullback it is enough to exhibit compatible lifts for the quasicategories $\mathcal{O}^\otimes$ and $N(\mathrm{Fin}_*^\times)$. Now the desired lifting property follows from the fact that α is initial $N(\mathrm{Fin}_*^\times)_{/(\langle n\rangle,S)}$ and that $\widehat{\alpha}$ is and that p-cocartesian edge in $\mathcal{O}^\otimes$ over α. $\square$

Corollary 8.0.15. Every simplicial sets map $F^0 : \mathcal{O}^\otimes \to \mathcal{C}$ admits a right Kan extension to $\mathcal{O}^\otimes \times_{\mathrm{Fin}_*} \mathrm{Fin}_*^\times$, and an arbitrary simplicial sets map $F : \mathcal{O}^\otimes \times_{\mathrm{Fin}_*} \mathrm{Fin}_*^\times \to \mathcal{C}$ is a right Kan extension of $F|_{\mathcal{O}^\otimes}$ if and only if $F(\widehat{\alpha})$ is an equivalence for every $\widehat{\alpha}$ as defined in Lemma 8.0.14

Proof. This follows immediately from Lemma 8.0.13 and Lemma 8.0.14. $\square$

Lemma 8.0.16. Define $\mathrm{Kan}'(\mathcal{O}^\otimes, \mathcal{C}^\times)$ as the subset of $\mathrm{sSets}(\mathcal{O}^\otimes \times_{N(Fin_*)} N(\mathrm{Fin}_*^\times), \mathcal{C})$ with the properties:

i) For every object $C \in (\mathcal{O}^\otimes)_{\langle n\rangle}$ and every subset $S \subset \langle n\rangle^\circ$, there is an equivalence

$$\prod \rho_i : F(C,S) \to \prod_{j\in S} F(C,\{j\}),$$

in the quasicategory $\mathcal{C}$.

ii) For every inert morphisms $(\varphi_\alpha) : C \to C'$ in $\mathcal{O}^\otimes$ covering $(\alpha) : \langle n\rangle \to \langle m\rangle$ and every $S \subset \langle m\rangle^\circ$, then the induced map $F(C, \alpha^{-1}S) \to F(C', S)$ is an equivalence.

And define $\mathrm{Kan}^{\mathrm{Lax}}(\mathcal{O}^\otimes, \mathcal{C}^\times)$ as the subset of $\mathrm{sSets}(\mathcal{O}^\otimes \times_{N(Fin_*)} N(\mathrm{Fin}_*^\times), \mathcal{C})$ with the properties:

a) The restriction $F|_{\mathcal{O}^\otimes} : \mathcal{O} \to \mathcal{C}$ is a Lax Cartesian map

b) F is a right Kan extension of $F|_{\mathcal{O}^\otimes}$.

Then $\mathrm{Kan}'(\mathcal{O}^\otimes, \mathcal{C}^\times) = \mathrm{Kan}^{\mathrm{Lax}}(\mathcal{O}^\otimes, \mathcal{C}^\times)$.

Proof. The equivalence between conditions ii) and b) is given by Corollary 8.0.15. Now, by definition of the embedding $\mathcal{O}^\otimes \hookrightarrow \mathcal{O}^\otimes \times_{N(\mathrm{Fin}_*)} N(\mathrm{Fin}_*^\times)$ and the definition of Lax Cartesian map, we have that explicitly condition a) is given by the following:

- Let C be an object of $\mathcal{O}^\otimes_{\langle n\rangle}$ and choose coCartesian lifts $\rho^!_{(n,i)} : C \to C_i$ of the maps $\rho_{(n,1)} : \langle n\rangle \to \langle 1\rangle$, then the induced map $\prod \rho^!_{(n,i)} : F(C, \langle n\rangle) \to \prod F(C_i, \{1\}^\circ)$ is an equivalence.

We see immediately that condition a) is equivalent to condition i). □

We are now ready to prove Proposition 8.0.11.

Proof of Proposition 8.0.11. Let $\bar{F} : \mathcal{O}^\otimes \to \mathcal{C}^\times$ be a map of simplicial sets over $N(\mathrm{Fin}_*)$. By definition this map is represented by a map $F : \mathcal{O}^\otimes \times \mathcal{O}'^\otimes \times_{N(Fin_*)} N(\mathrm{Fin}_*^\times) \to \mathcal{C}$. Moreover $\pi^*\bar{F}$ is given by the restriction

$$\begin{tikzcd} \mathcal{O}^\otimes \arrow[rr, "\pi^* F"] \arrow[d] & & \mathcal{C} \\ \mathcal{O}^\otimes \times_{N(Fin_*)} N(\mathrm{Fin}_*^\times) \arrow[urr, "F"'] & & \end{tikzcd},$$

Recall that $\mathrm{Alg}_{\mathcal{O}^\otimes}(\mathcal{C}^\times)$ is spanned by those simplicial sets maps $\mathcal{O}^\otimes \to \mathcal{C}^\times$ over $N(\mathrm{Fin}_*)$ preserving inert morphisms. By definition of $\mathcal{C}^\times$ and property 2 of Proposition 8.0.10, we have that

$$\mathrm{Alg}_{\mathcal{O}^\otimes}(\mathcal{C}^\times) \cong \mathrm{Kan}'(\mathcal{O}^\otimes, \mathcal{C}^\times).$$

Recall that Proposition 8.0.4 states there is an equivalence of quasicategories between the maps that are Kan extensions and those maps that have a colimit. Now by Proposition 8.0.4, together with Lemma 8.0.16, we have an equivalence of quasicategories

$$\mathrm{Kan}'(\mathcal{O}^\otimes, \mathcal{C}^\times) = \mathrm{Kan}^{\mathrm{Lax}}(\mathcal{O}^\otimes, \mathcal{C}^\times) \to \mathrm{PreKan}^{\mathrm{Lax}}(\mathcal{O}^\otimes, \mathcal{C}). \tag{8.1}$$

At last, by Corollary 8.0.15 every functor can be extended to a Kan extension, which implies $\mathrm{PreKan}(\mathcal{O}^\otimes, \mathcal{C}) = \underline{\mathrm{sSets}}(\mathcal{O}^\otimes, \mathcal{C})$ (where we are using the notation of Proposition 8.0.4). Therefore $\mathrm{PreKan}^{\mathrm{Lax}}(\mathcal{O}^\otimes, \mathcal{C}) = \underline{\mathrm{sSets}}^{\mathrm{Lax}}(\mathcal{O}^\otimes, \mathcal{C})$, which together with 8.1 concludes the proof. □

Proof of Proposition 8.0.12

In what follows we will consider $\mathcal{O}^\otimes \times \mathcal{O}'^\otimes$ as a full subquasicategory of $(\mathcal{O}^\otimes \times \mathcal{O}'^\otimes) \times_{N(Fin_*)} N(\mathrm{Fin}_*^\times)$ spanned by the objects of the form $(C, D; \langle nm\rangle^\circ)$ for $C \in \mathcal{O}^\otimes_{\langle n\rangle}$ and $D \in \mathcal{O}'^\otimes_{\langle m\rangle}$. For subsets $R \subset \langle n\rangle^\circ$ and $R' \subset \langle m\rangle^\circ$ we define the subset $R' \subset R$ defined by $(\alpha \wedge \beta)^{-1}(\langle |R||R|\rangle^\circ)$, where

$$\alpha(j) = \begin{cases} j & \text{if } j \in R \\ * & \text{otherwise,} \end{cases} \qquad \beta(i) = \begin{cases} i & \text{if } i \in R' \\ * & \text{otherwise.} \end{cases}$$

For example let $R = \{1, 2\} \subset \langle 3\rangle^\circ$ and $R' = \{1, 3\} \subset \langle 3\rangle^\circ$, then we have $R \wedge R' = \{1, 2, 7, 8\}$ and $R' \wedge R = \{1, 3, 4, 6\}$.

Lemma 8.0.17. With the assumptions of Proposition 8.0.12. Let (C, D, S) be an object of $\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times_{N(\mathrm{Fin}_*)} \mathrm{Fin}_*^{\times}$, where $C \in \mathcal{O}^{\otimes}_{\langle n\rangle}$, $D \in \mathcal{O}^{\otimes}_{\langle n'\rangle}$ and $S \subset \langle nn'\rangle^{\circ}$. Let $R \subset \langle n\rangle^{\circ}$ and $R' \subset \langle n'\rangle$ be the biggest subsets such that $R \wedge R' \subset S$. Define $\alpha : \langle n\rangle \to \langle R\rangle$ and $\beta : \langle n'\rangle \to \langle R'\rangle$ by

$$\alpha(j) = \begin{cases} j & \text{if } j \in R \\ * & \text{otherwise,} \end{cases} \qquad \beta(i) = \begin{cases} i & \text{if } i \in R' \\ * & \text{otherwise,} \end{cases}$$

let $C \to C'$ and $(\alpha^{-1}R) \to R$ and $(\beta^{-1}R') \to R'$ be (co)Cartesian lifts of α and β in $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ respectively. Then the induced 1-simplex $\widehat{\alpha \wedge \beta} : (C, D, S) \to (C', D', R \wedge R')$ is an initial object of $(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times_{N(\mathrm{Fin}_*)} \mathrm{Fin}_*^{\times})_{(C,D,S)/}$, where $(C', D', R \wedge R')$ is an object of $\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}$.

Proof. Following the same reasoning of 8.0.14 it is enough to proof that $\widehat{\alpha \wedge \beta}$ is an initial object of the subcategory of $(\mathrm{Fin}_*^{\times})_{(\langle nn'\rangle,S)/}$ that projects via $p : \mathrm{Fin}_*^{\times} \times \mathrm{Fin}_*^{\times} \to \mathrm{Fin}_* \times \mathrm{Fin}_* \to \mathrm{Fin}_*$ to $\mathrm{Fin}_{*\langle nn'\rangle/}$. It is not hard to see that this subcategory is the full category spanned by objects of the form $(\langle nn'\rangle, S) \xrightarrow{\eta\wedge\psi} (\langle mm'\rangle, S')$ for $\eta : \langle n\rangle \to \langle m\rangle$ and $\psi : \langle n'\rangle \to \langle m'\rangle$. We remark that in Lemma 8.0.14 we did not require to consider any subcategory, because in that case the subcategory in consideration was the whole category $(\mathrm{Fin}_*^{\times})_{(\langle n\rangle,S)/}$.

Now for any, $\phi \wedge \psi : (\langle nn'\rangle, S) \to (\langle mm'\rangle, S')$ in $\mathrm{Fin}_*^{\times}$ we have that $(\eta \wedge \psi)^{-1}(S') \subset S$ and is of the form $T \wedge T'$ for some $T \subset \langle n\rangle$ and $T' \subset \langle n'\rangle$, thus by definition of $R \wedge R'$ we must have that $(\phi \wedge \psi)^{-1}(S') \subset R \wedge R'$. This implies $(\langle R \wedge R'\rangle, R \wedge R')$ is an initial object in $(\mathrm{Fin}_*)_{(\langle nn'\rangle,S)/}$, and the explicit morphism is given by

$$\begin{tikzcd} (\langle nn'\rangle, S) \arrow[r, "\alpha"] \arrow[d, "\eta"'] & (\langle R \wedge R'\rangle, R \wedge R') \arrow[dl, dashed, "s"] \\ (\langle mm'\rangle, S') & \end{tikzcd} \qquad s(j) = \eta \wedge \psi(j), \text{ for } j \in RR' \subset \langle nn'\rangle.$$

□

Corollary 8.0.18. Every simplicial sets map $F^0 : \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{C}$ admits a right Kan extension to $\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times_{\mathrm{Fin}_*} \mathrm{Fin}_*^{\times}$, and an arbitrary simplicial sets map $F : \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times_{\mathrm{Fin}_*} \mathrm{Fin}_*^{\times} \to \mathcal{C}$ is a right Kan extension of $F|_{\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}}$ if and only if $F(\widehat{\alpha \wedge \beta})$ is an equivalence for every $\widehat{\alpha \wedge \beta}$ as defined in Lemma 8.0.17

Proof. This follows immediately from Lemma 8.0.13 and Lemma 8.0.17. □

Given the Lemma 8.0.17 and Corollary 8.0.18 the argument of the proof for Proposition 8.0.11 can be repeated *mutatis mutandis* to prove Proposition 8.0.12. We sketch the argument for completeness.

Lemma 8.0.19. Define $\mathrm{BiKan}'(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes} : \mathcal{C})$ as the subset of $\mathrm{sSets}(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \times_{N(Fin_*)} N(\mathrm{Fin}_*^{\times}), \mathcal{C})$ with the properties:

i) For every pair of objects $(C, D) \in (\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes})_{\langle n\rangle}$ and every subset $S \subset \langle n\rangle^{\circ}$, there is an equivalence
$$\prod \rho_i : F(C, D; S) \to \prod_{j \in S} F(C, D; \{j\}),$$
in the quasicategory $\mathcal{C}$.

ii) For every pair of inert morphisms $(\varphi_\alpha, \psi_\beta) : (C, D) \to (C', D')$ in $\mathcal{O}^\otimes \times \mathcal{O}'^\otimes$ covering $(\alpha \wedge \beta) : \langle nm \rangle \to \langle n'm' \rangle$ and every $S \subset \langle n'm' \rangle^\circ$, the induced map $F(C, D; (\alpha \wedge \beta)^{-1} S) \to F(C', D'; S)$ is an equivalence.

And define $\mathrm{BiKan}^{\mathrm{Lax}}(\mathcal{O}^\otimes, \mathcal{O}'^\otimes; \mathcal{C})$ as the subset of $\mathrm{sSets}(\mathcal{O}^\otimes \times \mathcal{O}'^\otimes \times_{N(Fin_*)} N(\mathrm{Fin}_*^\times), \mathcal{C})$ with the properties:

a) The restriction $F|_{\mathcal{O}^\otimes \times \mathcal{O}'^\otimes}$ is a Lax Cartesian bifunctor.

b) F is a right Kan extension of $F|_{\mathcal{O}^\otimes \times \mathcal{O}'^\otimes}$.

Then $\mathrm{BiKan}'(\mathcal{O}^\otimes, \mathcal{O}'^\otimes : \mathcal{C}) = \mathrm{BiKan}^{\mathrm{Lax}}(\mathcal{O}^\otimes, \mathcal{O}'^\otimes; \mathcal{C})$

Proof. Similar to the proof of Lemma 8.0.16. □

Proof of Proposition 8.0.12. Recall that $\mathrm{BiFunc}(\mathcal{O}^\otimes(\mathcal{O}^\times, \mathcal{O}'^\times; \mathcal{C}^\times)$ is spanned by those simplicial sets maps $\mathcal{O}^\otimes \times \mathcal{O}'^\otimes \to \mathcal{C}^\times$ over $N(\mathrm{Fin}_*)$ such that pairs of inert morphisms are mapped to inert morphisms. By definition of $\mathcal{C}^\times$ and property 2 of Proposition 8.0.10, we have that

$$\mathrm{BiFunc}(\mathcal{O}^\otimes, \mathcal{C}^\times) \cong \mathrm{BiKan}'(\mathcal{O}^\otimes, \mathcal{O}'^\otimes, \mathcal{C}^\times).$$

Then, the same arguments as in the proof of Proposition 8.0.11 gives an equivalence of quasicategories

$$\mathrm{BiKan}'(\mathcal{O}^\otimes, \mathcal{O}'^\otimes : \mathcal{C}) = \mathrm{BiKan}^{\mathrm{Lax}}(\mathcal{O}^\otimes, \mathcal{O}'^\otimes; \mathcal{C}) \to \mathrm{PreKan}^{\mathrm{Lax}}(\mathcal{O}^\otimes \times \mathcal{O}'^\otimes, \mathcal{C}).$$

and by 8.0.18 we have that $\mathrm{PreKan}^{\mathrm{Lax}}(\mathcal{O}^\otimes \times \mathcal{O}'^\otimes, \mathcal{C}) = \underline{\mathrm{sSets}}^{\mathrm{BiLax}}(\mathcal{O}^\otimes, \mathcal{C})$, which concludes the proof. □

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